

1994

Dynamic stability of a self-excited elastic beam

Francisco L. Ziegelmuller

Follow this and additional works at: <http://scholarworks.rit.edu/theses>

Recommended Citation

Ziegelmuller, Francisco L., "Dynamic stability of a self-excited elastic beam" (1994). Thesis. Rochester Institute of Technology.
Accessed from

This Thesis is brought to you for free and open access by the Thesis/Dissertation Collections at RIT Scholar Works. It has been accepted for inclusion in Theses by an authorized administrator of RIT Scholar Works. For more information, please contact ritscholarworks@rit.edu.

DYNAMIC STABILITY OF A SELF-EXCITED ELASTIC BEAM

Francisco L. Ziegelmuller

**A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of
Master of Science
in
Mechanical Engineering**

Approved by: Professor J.S. Torok (Thesis Advisor)

Professor R. G. Budynas

Professor R. H. Hetnarski

Professor C. W. Haines (Department Head)

Department of Mechanical Engineering
College of Engineering
Rochester Institute of Technology
Rochester, New York

July 1994

PERMISSION GRANTED:

I, Francisco L. Ziegelmuller, hereby grant permission to the Wallace Memorial Library of the Rochester Institute of Technology to reproduce my thesis entitled "Dynamic Stability of A Self-Excited Elastic Beam" in whole or in part. Any reproduction will not be for commercial use or profit.

July 15, 1994

Francisco L. Ziegelmuller

ABSTRACT:

An elastic cantilever beam under pressure contact with a moving web undergoes self-excited vibration that may lead to unstable and even self destructive behavior.

The equations of motion for the system are derived from the principle of virtual work and Hamilton's principle using the techniques of the calculus of variations. The beam, being a continuum with infinite degrees of freedom, is approximated by a model with a finite number of degrees of freedom using Galerkin's method. The characteristic equation for the model is examined to determine its dynamic criterion for stability. A parametric study is performed to determine the effects of the beam properties such as beam length (L), extension (W), thickness (h), elastic modulus (E), stiffness (EI), beam inclination angle (θ) with respect to moving web, the static and kinematic coefficient of friction (μ_s, μ_k). The beam response due to the motion of the contacting web is undertaken to evaluate critical properties to be used as guide in the design of stable beam for such applications.

TABLE OF CONTENTS

| | |
|------------|--|
| [1] | Introduction |
| [2] | Background |
| 2.1 | Introduction |
| 2.2 | Principle of Virtual Displacements |
| 2.3 | Energy Method using Hamilton's Principle |
| 2.4 | Structural (Static) Stability |
| 2.5 | Dynamic Stability |
| [3] | The Governing Dynamical System |
| 3.1 | Derivation of Governing Equations |
| 3.2 | Modal Analysis of A Cantilevered Beam |
| 3.3 | Numerical Values of Beam Parameters and Constants |
| 3.4 | Governing Equations |
| [4] | Dynamical Stability Analysis |
| 4.1 | Bifurcation and Eigenvalue Approaches |
| 4.2 | Solution of Characteristic Equation |
| 4.3 | Dynamic Stability Criterion |
| 4.4 | Parametric Evaluation of Stability |
| [5] | Conclusions |

LIST OF SYMBOLS

| | |
|-----------------------------|--|
| x | coordinate along the beam length axis |
| t | time coordinate |
| q_i | generalized coordinates ($i = 1, 2, \dots, N$) |
| N | number of modes or degrees of freedom |
| DOF | degrees of freedom |
| NDOF | number of degrees of freedom |
| $\partial q_i / \partial t$ | partial derivative of q_i with respect to t |
| $\partial q_i / \partial x$ | partial derivative of q_i with respect to x |
| $Q_i, p(t)$ | generalized force |
| $u_i(t)$ | axial generalized displacement |
| $v_i(t)$ | transverse generalized displacement |
| $u(x, t)$ | beam axial deformation as a function of x and t |
| $v(x, t)$ | beam transverse deflection as a function of x and t |
| u', u'', \dots | partial derivatives of $u(x, t)$ with respect to x ($\partial u / \partial x$, $\partial^2 u / \partial x^2$, ... or u_x, u_{xx}, \dots) |
| v', v'', \dots | partial derivatives of $v(x, t)$ with respect to x ($\partial v / \partial x$, $\partial^2 v / \partial x^2$, ... or v_x, v_{xx}, \dots) |
| u_t, u_{tt}, \dots | partial derivatives of $u(x, t)$ with respect to t ($\partial u / \partial t$, $\partial^2 u / \partial t^2$, ...) |
| v_t, v_{tt}, \dots | partial derivatives of $v(x, t)$ with respect to t ($\partial v / \partial t$, $\partial^2 v / \partial t^2$, ...) |
| δ | variational operator or virtual operator |
| T | total kinetic energy of the system |
| V | total potential energy of the system |
| U_a | elastic or strain energy of the beam in tension or compression |

| | |
|-------------------|---|
| U_b | elastic or strain energy of the beam in bending |
| W_{cons} | work done by external conservative forces ($= -V$) |
| W_{nc} | work done by external nonconservative forces |
| W | combined work by the real and inertial forces |
| $\phi, \phi(x)$ | transverse displacement of the beam |
| $\phi_i(x)$ | characteristic modal function of a uniform cantilevered beam (mode shape) |
| $\phi'_i(x)$ | partial derivative of $\phi_i(x)$ with respect to x ($\partial\phi_i/\partial x$) |
| ω_i | natural frequency corresponding to ϕ_i |
| λ_i | spatial frequency constant |
| α_i | constant in the mode shapes of a cantilevered beam |
| $\psi, \psi(x)$ | axial displacement of the beam |
| $\psi_i(x)$ | characteristic modal function of a uniform cantilevered beam (mode shape) |
| $\psi'_i(x)$ | partial derivative of $\psi_i(x)$ with respect to x ($\partial\psi_i/\partial x$) |
| E | Young's modulus or elastic modulus |
| I | moment of inertia of the beam |
| $E \cdot I$ | flexural rigidity or bending stiffness |
| L | beam length |
| h | beam thickness |
| b | beam width |
| A | cross-sectional area of the beam |
| ρ | mass per unit length of the beam |
| ω | forcing frequency |
| μ | coefficient of friction between the beam and web |
| μ_s | static coefficient of friction |
| μ_k | kinematic coefficient of friction |

| | |
|--------------------|---|
| θ | inclination angle between the beam and the moving web or inclination angle of the undeflected beam with the tangential line at the point of contact with the web |
| N | normal force applied by the web on the beam |
| T_s | web tension |
| f | frictional force, equals to $\mu_s \cdot N$ up to the onset of motion and equals to $\mu_k \cdot N$ afterwards. Its direction is always opposite to the motion. |
| M | moment at the free end of the beam |
| f_x, f_y, M | x- and y- components of N and f , and M magnitudes |
| ζ | angle defined between N and the blade surface facing the web |
| α | attack angle of the beam on the roller |
| β, γ | angles defined by the deflected web |
| η | angle included between tension forces ($=180 - \beta - \gamma$) |
| s, s_1 | web span between rollers and beam |
| s | a root to the characteristic equation which equals $\sigma \pm j\omega$. σ is the real component and ω is the imaginary component. |
| e | web deflection |
| r, e | roller radius and roller excentricity |
| v_w | web velocity |
| v_b | velocity of the free end of beam in contact with web ($= \{ [u_t(L,t)]^2 + [v_t(L,t)]^2 \}^{.5}$) |
| s_v | slip velocity ($= v_w - v_b$) |
| $\text{sgn}(x)$ | sign of x |
| c | damping coefficient |
| σ, ϵ | stress and strain ($\sigma = E \cdot \epsilon$) |

[1] Introduction

The application of a cantilever beam in pressure contact with a moving web results in nonlinear vibrations due to self-excited motion or a condition commonly referred to as "slip-stick motion". Such vibrations can reach levels that lead to unstable behavior resulting in a condition referred to as "tuck-under", where the "free" end of the beam experiences an inversion in its deformed shape (figure 1). "Tuck-under" may result in self-destructive behavior of the beam and/or web.

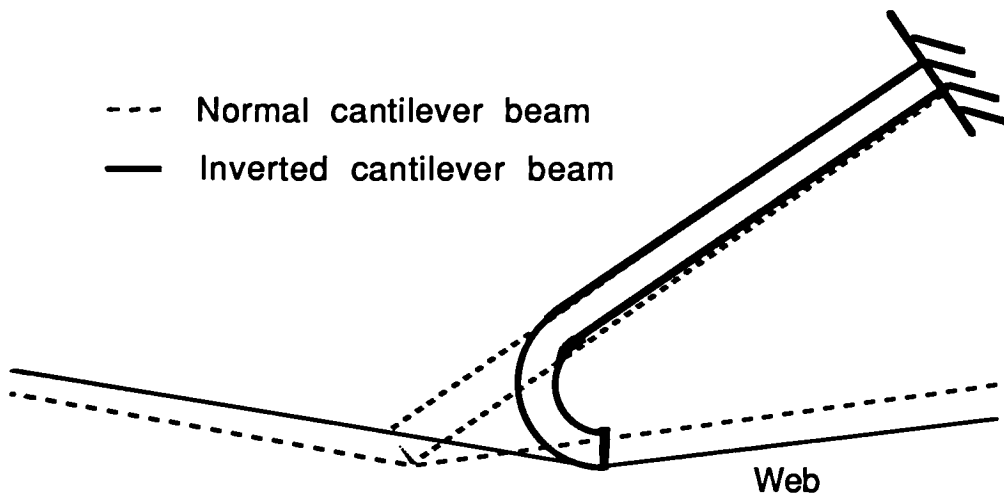


Figure 1

In actual applications, the beam is in reality a plate, commonly referred to as a "doctor blade". It is usually made of polymeric or metallic material and it is used in copiers to clean a moving web of fine powder. In the absence of such fine powder, the friction force increases causing the blade to change its deformed shape. If a critical load is reached, the blade may tuck under the web. This behavior generally occurs locally and then rapidly expanding along the blade width.

The above behavior can be repeated for shorter plate width and hence, the author has opted for an analysis simplification using a

cantilever beam as a model. The analysis is performed by first presenting two versions of the problem and then by describing the problem as one in which the beam is inclined from the web by a given angle. The normal and frictional forces will be defined with respect to this angle. Then the equations of motion for the system are derived from the principle of virtual work and Hamilton's principle using the techniques of the calculus of variations. The beam, being a continuum with infinite degrees of freedom, is approximated by a model with a finite number of degrees of freedom using Galerkin's method. The characteristic equation for the model is examined to determine its dynamic criterion for stability. The goal of this investigation is a parametric study of the effects of the beam properties such as beam length (L), extension (W), thickness (h), elastic modulus (E), stiffness (EI), beam inclination angle (θ) with respect to moving web, static and kinematic coefficient of friction (μ_s, μ_k). The beam response due to the motion of the contacting web is analyzed to evaluate critical properties to be used as guide in the design of stable beam for such applications.

The first problem presented examines the case when the cantilever beam engages the web over a relatively hard roller (figure 2) so that the normal force, N , is in the radial direction and the friction force, f , perpendicular to the normal force and related to it by the expression: $|f|=|\mu \cdot N|$, where μ is the coefficient of friction between the contacting surfaces. The beam is inclined from the tangential line at the point of contact by an angle θ . The beam contacts the web at an attack angle α along the roller. These forces can be decomposed along x- and y-axis components as:

$$f_x = N[\sin\theta + \mu\cos\theta] \quad \text{and} \quad f_y = N[\cos\theta - \mu\sin\theta]. \quad (1.1)$$

**(A) CANTILEVER BEAM WITH FREE END IN PRESSURE CONTACT
WITH A MOVING WEB OVER A HARD ROLLER**

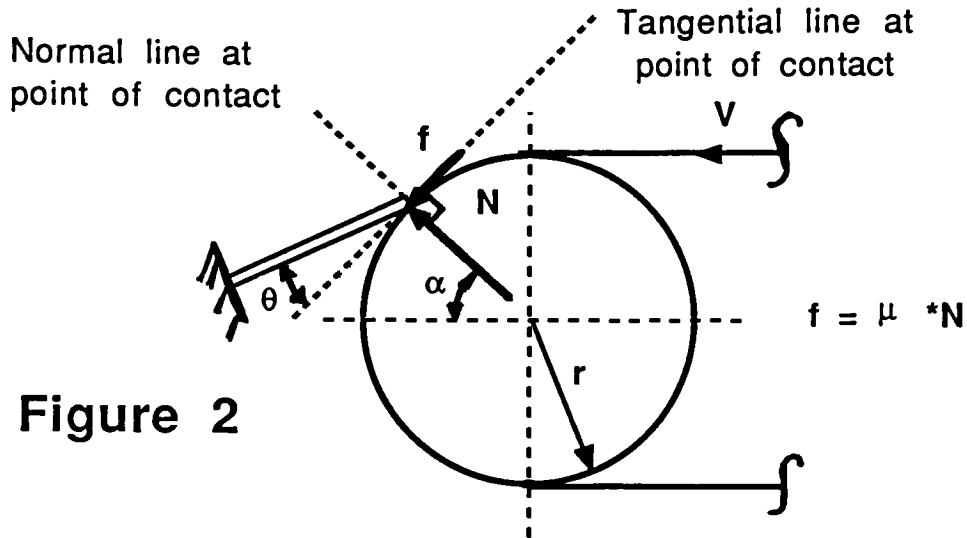


Figure 2

Due to the fact that f_x is an eccentric load, we can replace it by a load f_x acting along the neutral axis and a counterclockwise moment M acting along the neutral axis at the free end of the beam (figure 3). The magnitude of this moment is:

$$M = f_x \cdot h/2 = [N \cdot h/2] \cdot [\sin \theta + \mu \cdot \cos \theta] \quad (1.2)$$

The beam has rectangular cross-section with the following dimensions: length L , width b , thickness h , and moment of inertia I which equals $b \cdot h^3/12$. The materials to be investigated will be stainless steel and polyurethane with the following values for their respective Young's modulus, E : $29 \cdot 10^6$ and 1000 psi. The coefficient of friction values to be studied range from 0.3 to 1.5. The loads will be given in pounds/linear inch of beam width. The normal load N range to be investigated is from 0.1 to 0.2 lb/in of width.

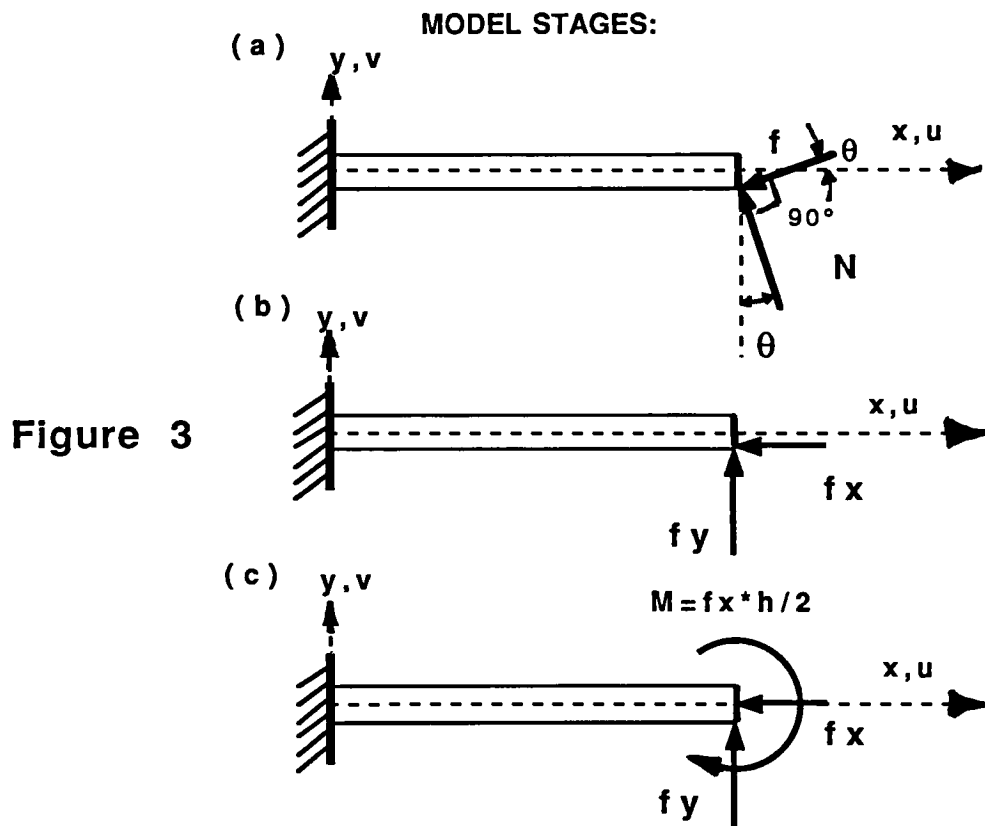
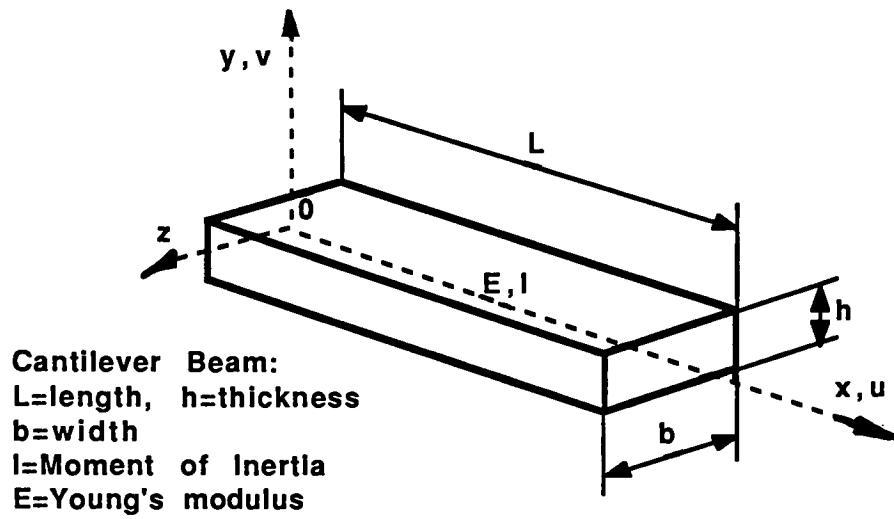


Figure 3

The web moves with an approximately constant linear velocity v_w . The drive roller radius is r and it may have a small eccentricity e which leads to an oscillating deflection of the cantilever free end (**figure 4**).

The second version of the problem places the cantilever beam in pressure contact with the moving web in a section between two hard rollers (**figure 5**). One of the rollers is spring loaded to provide enough tension for the web to be friction driven by fixed roller. The beam loads the web causing it to deflect by e . The span between the rollers is s . The beam contacts the web at a distance s_1 from the spring loaded roller. In this case the load N will be in the direction of the resultant of the web tension components, T_s , as shown in **figure 6**. The friction force will be perpendicular to the load N . The beam is inclined from the undeflected web by an angle θ . The angle included between the tension forces is:

$$\eta = 180 - \beta - \gamma, \quad (1.3)$$

where:

$$\gamma = \tan^{-1}[e/(s-s_1)] \quad \text{and} \quad \beta = \tan^{-1}(e/s_1). \quad (1.4)$$

The load N makes an angle equal to half η with the tension force. The beam makes an angle equal to:

$$\zeta = \eta/2 - \theta + \gamma \quad (1.5)$$

with the beam surface facing the web. But,

$$\zeta = 90^\circ - \beta/2 - \gamma/2 - \theta + \gamma = 90^\circ - \beta/2 + \gamma/2 - \theta. \quad (1.6)$$

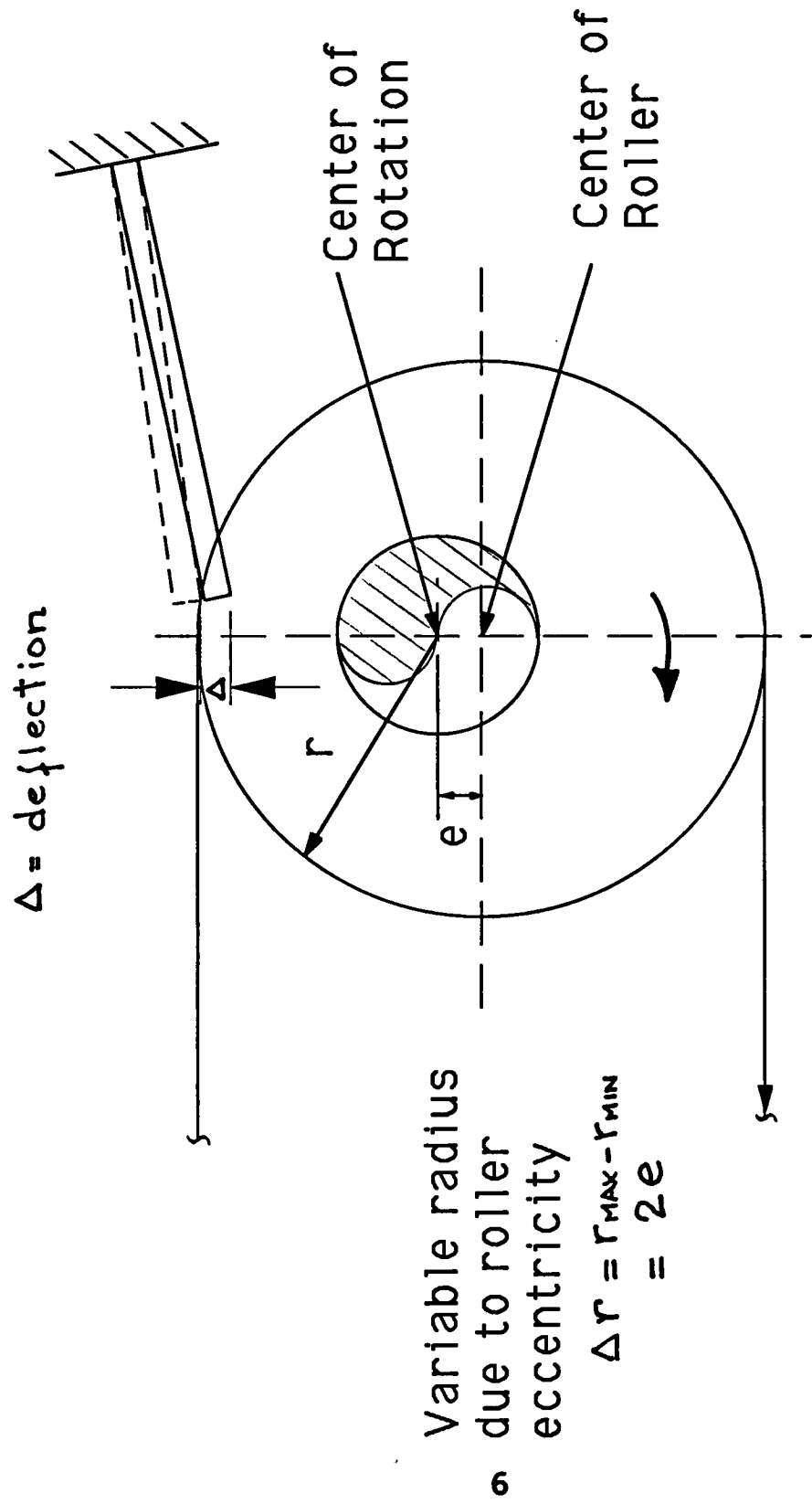


FIGURE 4

**(B) CANTILEVER BEAM WITH FREE END IN PRESSURE CONTACT
WITH A MOVING WEB WITHOUT A HARD BACK-UP ROLLER**

Web tension is controlled by a spring with a load rate of k

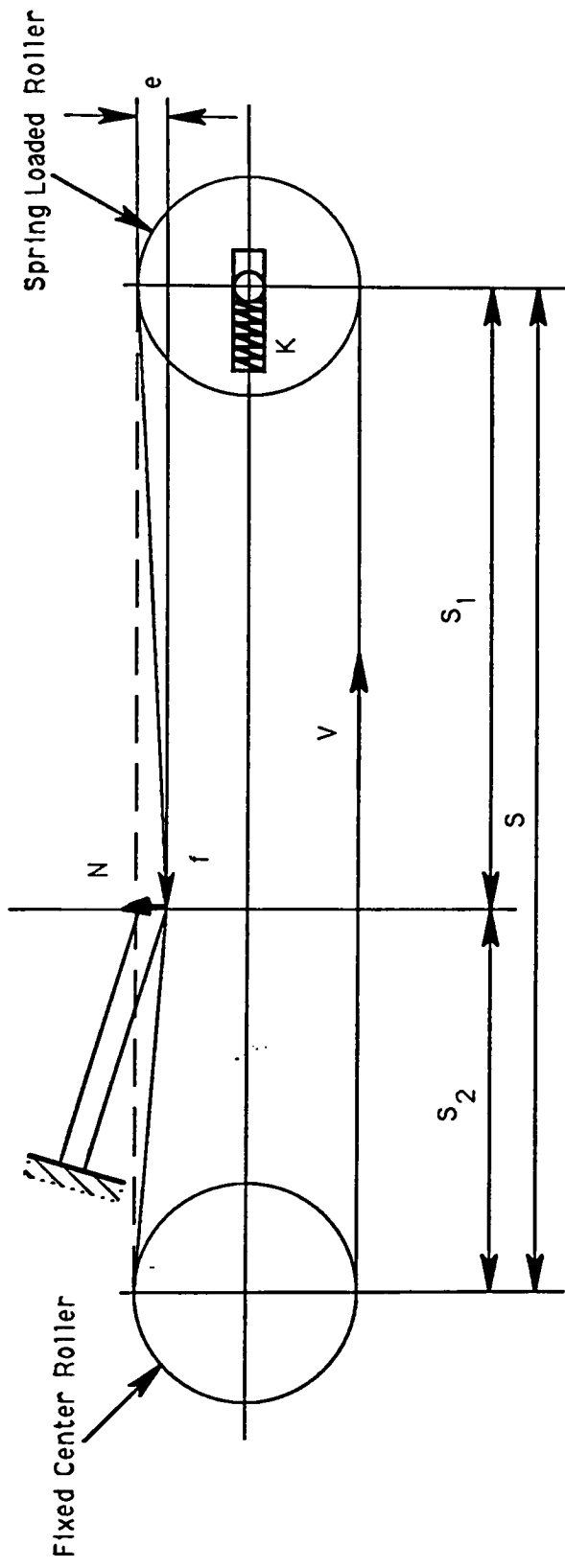


Figure 5

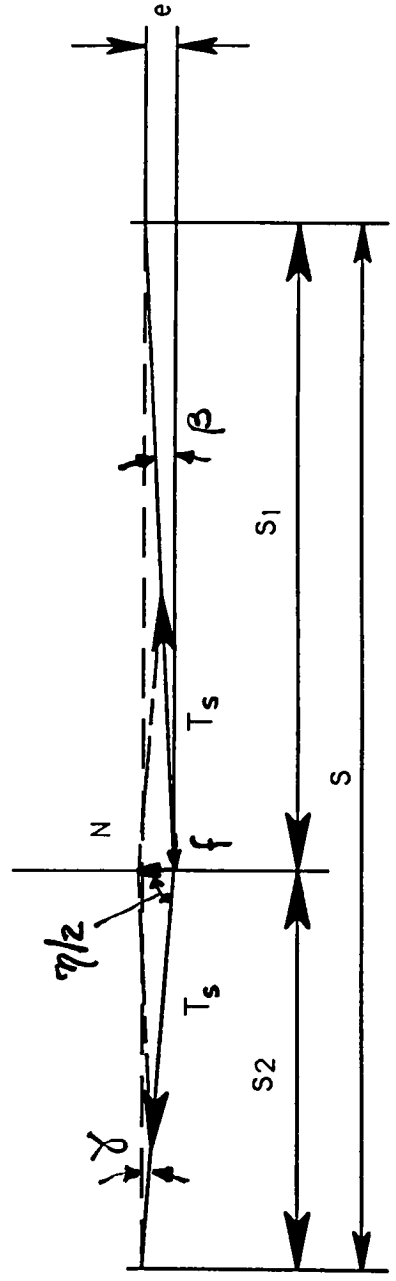


Figure 6

This geometric configuration is shown in **figure 7**. Notice that if β is equal to γ , that is, the beam contacts the web at midspan and/or these angles are very small in magnitude, then we would end up with the same geometric configuration of the first version of the problem.

In summary, we will be investigating a cantilever beam in pressure contact with the web under a normal load N , making an angle ζ with the blade surface facing the web and a frictional force f perpendicular to N . From **figure 8**, we can derive the x- and y- components of the forces and the end moment as:

$$f_y = N \cdot \sin \zeta - f \cdot \sin(90^\circ - \zeta) = N \cdot (\sin \zeta - \mu \cdot \cos \zeta), \quad (1.7)$$

$$f_x = N \cdot \cos \zeta + f \cdot \cos(90^\circ - \zeta) = N \cdot (\cos \zeta + \mu \cdot \sin \zeta), \text{ and} \quad (1.8)$$

$$M = f_x \cdot h/2 = (N \cdot h/2) \cdot (\cos \zeta + \mu \cdot \sin \zeta). \quad (1.9)$$

The latter version of the problem is more generalized, but usually the web angles β and γ are small. We will assume they are negligible for the analysis of the cantilever beam.

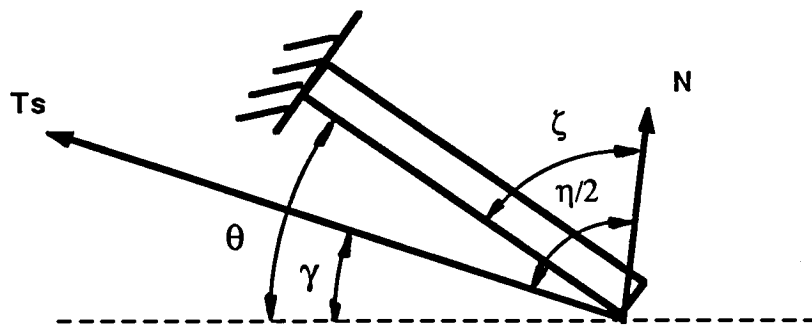


Figure 7

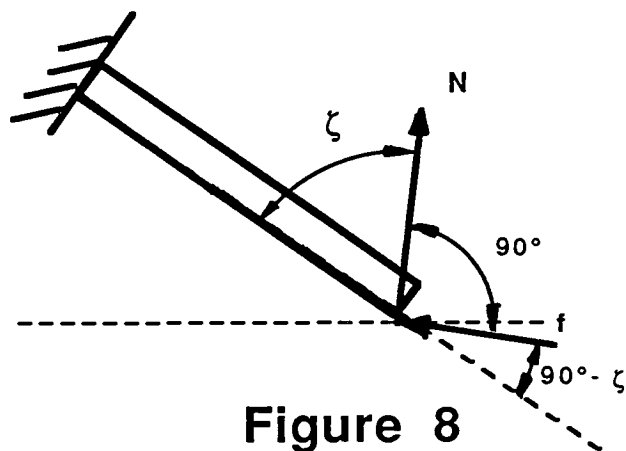


Figure 8

[2] Background

2.1 Introduction

The cantilever beam problem presented in the previous section has very low transverse stiffness and high axial stiffness. Runout or eccentricity of the roller, variation in frictional forces due to dry web condition and differential levels of lubrication will result in harmonic forcing functions to be applied at the free end of the beam. These various loading conditions may result in unstable behavior of the beam.

Elastic beam stability has been extensively studied as models for such problems as a flexible missile under an end thrust [1], buckling of a magnetoelastic system [2], elastic stability under follower forces [3] such as a structural part of an aircraft under aerodynamic loads, a cantilever beam conveying fluid which loses stability by flutter, and others. The study of the dynamic stability of an elastic beam under an harmonically varying load with a pinned end and an end constrained to move axially has been presented by Mettler, Timoshenko, Bolotin and others [4,5,6].

Approximate techniques using numerical methods are often required to solve these types of problems. Galerkin's technique was first applied by Leipholz (1962a) to calculate bifurcation loads for a nonconservative system [1,7]. Timoshenko [5], Mettler [4] and Bolotin [6] have used Galerkin's method to convert the equations of motion to time differential equation which are studied for their stability using Floquet theory [8,9,10,11,12]. Levinson [7] showed that the conventional Hamilton's principle with nonconservative forces may be treated as a variational problem with some constraints, and then the Ritz method can be employed. Leipholz used the adjoint system to establish a well posed variational

problem without any constraint conditions. The finite element method has been used based on the variational approach suggested by Levinson, and based on Leipholz's variational principle. The ordinary finite difference method was used with the concept of a transfer matrix to solve a nonconservative problem by Leipholz. Guran and Ossia investigated the dynamic stability of a uniform free-free rod under an end thrust using finite difference techniques with the concept of a transfer matrix. In nonconservative stability problems, the dynamic stability criteria must be applied. The bifurcation or critical load is the smallest load which, when slightly disturbed, causes a change in the equilibrium configuration of the system. A forced vibration analysis must be performed to determine this load.

The derivation of the governing partial differential equations of motion and the evaluation of critical loads by using Hamilton's principle follows the work presented by Guran, Ossia [1] and Levinson [7].

Based on a set of generalized coordinates, q_i , the equations of motion are determined from variational principles. Virtual displacements and virtual work concepts are utilized in the evaluation of energy transformations. The beam transverse deflection, $v(x,t)$ is approximated by the sum of the N lowest mode shapes of a uniform beam under free vibration [13]. The beam deflection may then be written as:

$$v(x,t) = \sum_{i=1}^N \phi_i(x) * v_i(t) \quad (2.1)$$

where N = number of degrees of freedom, nodes or mode shapes
 $\phi_i(x)$ = mode shape associated with node i
 $v_i(t)$ = generalized displacement

The above expression is then used in association with Hamilton's principle to derive the equations of motion of the forced system and its natural and geometric boundary conditions. The equations of motion are then manipulated to arrive at the characteristic equation for the system by using Galerkin's method. Roots of the characteristic equation are analyzed using the dynamic criterion for stability to determine the critical or bifurcation load. This load will then be used in a parametric evaluation of stability for the cantilever beam.

2.2 Principle of Virtual Displacements and Virtual Work

The application of virtual displacements is used to define energy transformations in a system. This technique is used to approximate continuous bodies by modelling a system with an infinite number of degrees of freedom (DOF) by one having only a finite number of degrees of freedom [13].

A change in the configuration of a system is specified by a displacement coordinate. The system may be under a certain number of constraints which are kinematical restrictions on possible configurations that the system may undertake. A virtual displacement is an imaginary infinitesimal change of configuration of a system conforming to its constraints. An arbitrary configuration of a system may be described using a set of generalized coordinates which are linearly independent displacements, consistent with the constraints imposed on the system. The symbol q_i (where $i=1,2,...,N$) are used for the generalized coordinates of an N-DOF system. A virtual displacement is then denoted by δq_i .

The virtual work, δW , is the work of the forces acting on a system as it undertakes a set of admissible virtual displacements:

$$\delta W = \sum_{i=1}^N Q_i \delta q_i \quad (2.2)$$

where Q_i = the generalized force

δq_i = virtual displacement by Q_i

Q_i is the virtual work done when $\delta q_i=1$ and $\delta q_j=0$ for $j \neq i$. This concept of generalized force is used extensively in structural dynamics and in the study of structural stability.

The principle of virtual displacements may be stated as:

"For any arbitrary virtual displacement of a system in equilibrium, the combined virtual work of real forces and inertia forces must vanish". This may be represented as:

$$\delta W \equiv \delta W_{\text{real forces}} + \delta W_{\text{inertia forces}} = 0 \quad (2.3)$$

where δW = the sum of the virtual work done by real forces and inertia forces.

The principle of virtual displacements can produce a generalized parameter model of a continuous system in a manner that approximates its flexible behavior. This procedure is commonly referred to as the assumed-modes method. A continuous system is one whose deformation is described by one or more functions of one, two or three spatial variables and time. The deformation of a cantilever beam in figure 6 is specified in terms of the deflection curve $v(x,t)$ of the neutral axis. Kinematical constraints placed on the displacement and/or slope on portions of the boundary of the continuous body are called a geometric boundary conditions (GBC). A virtual displacement of a continuous system is an

imaginary infinitesimal change in the displacement functions with all GBC enforced. For the cantilever beam, the GBC are:

$$v(0,t) = v'(0,t) = 0 \quad (2.4)$$

where $v' = \partial v / \partial x$

The dotted curve in figure 9 shows a possible virtual displacement, $\delta v(x,t)$, of the beam. The only conditions on $\delta v(x,t)$ is that it satisfies the homogeneous form of the same GBC as $v(x,t)$. That is:

$$\delta v(0,t) = \delta v'(0,t) = 0. \quad (2.5)$$

$\delta v(x,t)$ is not a function of time, but it is to be understood as an arbitrary small change of configuration, relative to the configuration of the beam at time t .

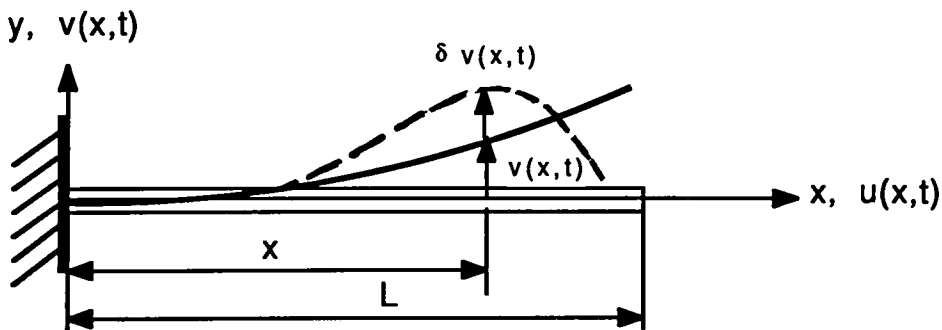


Figure 9:
Cantilever Beam, GBC, virtual displacements

An admissible function is a function that satisfies the GBC of the system under evaluation and that possesses derivatives of an order at least equal to that appearing in the strain energy expression for the

system. An assumed-mode or shape function is an admissible function that is used to approximate the deformation of a continuous system. To approximate a continuous system with a generalized-parameter N-DOF model, N assumed-modes are used. The deflection of the beam may be approximated by:

$$v(x,t) = \sum_{i=1}^N \phi_i(x) * v_i(t) \quad (2.1)$$

Any admissible function may be used as $\phi_i(x)$, but a shape that closely resembles the deformed shape of the beam should be selected. $v_i(t)$ is called a generalized displacement for the N-DOF system. It is determined as the solution to an ordinary differential equation.

The principle of virtual displacements is employed here to create a N-DOF generalized-parameter model based on (2.1). For a continuous system, it is convenient to introduce potential energy, that is, the work done by conservative forces.

$$\delta W_{\text{real forces}} = \delta W_{\text{cons}} + \delta W_{\text{nc}} \quad (2.6)$$

where δW_{cons} is the virtual work of conservative forces and δW_{nc} is that of nonconservative forces.

From the definition of a conservative force:

$$\delta W_{\text{cons}} = -\delta V \quad (2.7)$$

where δV is the change in the potential energy as the conservative forces move through a virtual change in configuration, for example, $\delta v(x,t)$ for a beam. Consequently, we can write that

$$\delta W = \delta W_{nc} - \delta V + \delta W_{inertia\ forces} = 0 \quad (2.8)$$

Strain energy in a beam undergoing pure axial deformation $u(x,t)$ and in a beam undergoing pure transverse deflection $v(x,t)$ may be expressed respectively as:

$$U_a = .5 \int_0^L A E (u')^2 dx \quad (2.9)$$

and

$$U_b = .5 \int_0^L E I (v'')^2 dx \quad (2.10)$$

where

U_a , U_b are the axial and the transverse strain energy respectively

A is the cross-sectional area of the beam

E is the elastic modulus of the beam material

I is the moment of inertia of the beam about the bending axis

u' is the first partial derivative of $u(x,t)$ with respect to x or $\partial u / \partial x$

v'' is the second partial derivative of $v(x,t)$ with respect to x or $\partial^2 v / \partial x^2$.

Consequently, the variational operator δ applied to the above expressions result in:

$$\delta U_a = \int_0^L (A E u') \delta u' dx \quad (2.9-a)$$

and

$$\delta U_b = \int_0^L (E \cdot I \cdot v'') \cdot \delta v'' \cdot dx \quad (2.10-a)$$

If the beam is deformed in both the axial and transverse direction as shown in **figure 10**, the strain along the neutral axis can be written as {Appendix):

$$\epsilon = u' + .5 \cdot v'^2. \quad (2.11)$$

The strain energy along the neutral axis may then be written as:

$$\begin{aligned} U_a &= \frac{1}{2} \int_V \sigma \cdot \epsilon \cdot dV = \frac{1}{2} \int_0^L \sigma \cdot \epsilon \cdot A \cdot dx = \frac{1}{2} \int_0^L E \cdot A \cdot \epsilon^2 \cdot dx \\ &= \frac{1}{2} \int_0^L E \cdot A \cdot (u' + .5 \cdot v'^2)^2 \cdot dx \quad (2.12) \end{aligned}$$

where σ and ϵ are the axial stress and axial strain respectively and V represents the volume of the beam.

Applying the variational operator to the above expression, we get:

$$\delta U_a = \int_0^L E \cdot A \cdot (u' + .5 \cdot v'^2) \cdot (u' + .5 \cdot v'^2)' \cdot dx \quad (2.12-a)$$

We assume in these derivations that the beam material is linearly elastic, homogeneous, of constant cross section and that it follows Hooke's law relating stress to strain as $\sigma = E \cdot \epsilon$. Planes of deformation along the longitudinal axis will remain perpendicular to the neutral axis {no twisting}.

STRAIN ALONG THE NEUTRAL AXIS

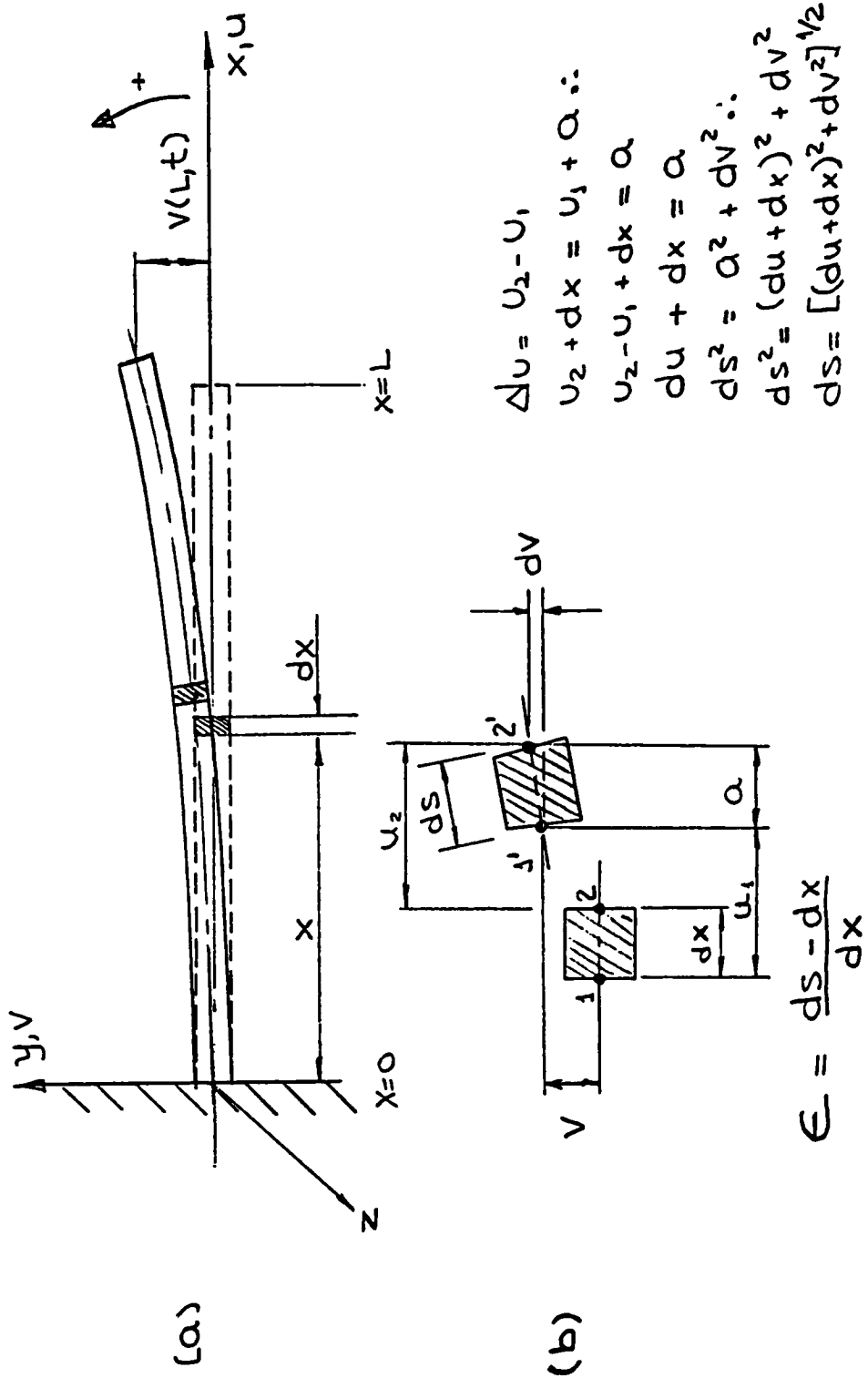


FIGURE 10

The damping of a beam may be the result of internal damping in the material and Coulomb, or dry friction, damping between the beam free end and the moving web. These types of damping generally results in nonlinear behavior of the beam.

The internal damping of a beam may be modelled as a viscous type of damper. If we let c represent the viscous damping coefficient, then the work done by this nonconservative damping force, $c*v_t$, is:

$$W_{nc} = \int_0^L c*v_t*v(x,t)*dx \quad (2.13)$$

where $v_t = \partial v(x,t)/\partial t$.

Similarly, we get:

$$\delta W_{nc} = \int_0^L c*[v_t*\delta v(x,t) + v(x,t)*\delta v_t]*dx \quad (2.14)$$

An axial damping force could also be added to the beam model and its strain energy would be arrived at by substituting u for v in the above expressions. In this analysis, we will neglect the structural damping of the beam. Mettler [4] provided some insight into the damping on the structural stability of the beams.

The frictional force always opposes motion. When there is slip between the beam and the moving web, the frictional force is opposite to that of the slip velocity s_v [14,15]. The slip velocity may be defined as:

$$s_v = v_w - v_b, \quad (2.15)$$

where v_w = web velocity

and v_b = velocity of the free end of the beam.

The velocity of the free end of the beam may be expressed as:

$$v_b = \{ [u_t(L,t)]^2 + [v_t(L,t)]^2 \}^{.5} \quad (2.16)$$

where u_t and v_t are the partial derivatives of u and v with respect to t .

The coefficient of friction between sliding surfaces may be expressed as:

$$\mu = \mu_s - k_1(v_w - v_b) = \mu_s - k_1 s_v \quad (2.17)$$

where μ_s = static coefficient of friction

k_1 = a constant, usually a small number

The frictional force may then be expressed as:

$$f = \mu \cdot N \cdot \text{sgn}(s_v) \quad (2.18)$$

where $\text{sgn}(s_v)$ means the sign of the slip velocity.

We will assume in the analysis that the friction force will not vary with the slip velocity to simplify the model. The frictional force will be assumed to be constant with a small perturbation added to it. This perturbation may be assumed to vary harmonically with time and with a frequency ω which is much smaller than the fundamental frequency of the longitudinal or axial vibrations of the beam.

2.3 Energy Method using Hamilton's Principle

Newton's laws and the principle of virtual work are employed to derive the equations of motion of mechanical systems. Hamilton's principle, which is directly related to the principle of virtual displacements, employs kinetic and potential energy, which are scalar quantities, rather than the virtual work of inertia forces and elastic forces to derive the equations of motion. For continuous systems, the use of Hamilton's principle provides an additional bonus which are the natural boundary conditions.

In the formulation of Hamilton's principle, virtual displacements, or virtual changes of configuration, are used. The virtual change of configuration must satisfy all geometric boundary conditions. Hamilton also assumed that the configuration is specified at times t_1 and t_2 . For the cantilever beam in figure 9, this would imply that:

$$\delta v(x, t_1) = \delta v(x, t_2) = 0. \quad (2.19)$$

Hamilton's principle may be stated as:

$$\int_{t_1}^{t_2} \delta(T - V) * dt + \int_{t_1}^{t_2} \delta W_{nc} * dt = 0 \quad (2.20)$$

where

T = total kinetic energy of the system

V = potential energy of the system, including the strain energy and the potential energy of conservative external forces

δW_{nc} = virtual work done by nonconservative forces, including damping forces and external forces not accounted for in V .

t_1, t_2 = times at which the configuration of the system is known.

2.4 Structural (Static) Stability

Structural stability relates to the study of stability of structures under static equilibrium. Hence, there is no change in kinetic energy in the system or the inertial forces are neglected ($\delta T = 0$).

This method is valid only for structural systems that are elastic and conservative and members are assumed to be geometrically perfect. For a geometrically perfect member, the lateral deflections of a centrally loaded beam subjected to in-plane forces will not occur until the applied load reaches a critical value. At this critical load value, a small disturbance on the beam will generate a large lateral deflection [16,17].

To obtain the critical load, we first have to arrive at a differential equation for the beam in a slightly deformed state. The solution to the characteristic equation derived from this governing differential equation yields the critical load of the beam[16].

There are two ways to derive the governing equations for the structure: the vector approach (Newton's Laws) and the energy method. Both methods will lead to eigenvalue, i.e. bifurcation, analysis. We will utilize the energy method approach.

The total potential energy of a system, Π , is:

$$\Pi = U + V \quad (2.21)$$

where

U = internal strain energy

V = potential energy of applied loads = $-W$

W = work done by external loads.

Thus,

$$\Pi = U - W \quad (2.22)$$

If we restrict the analysis to elastic systems subjected by conservative forces, we can apply the principle of stationary potential energy which can be developed from the principle of virtual work [16]. The principle of stationary potential energy states that:

"A structure is in (static) equilibrium if and only if the total potential energy is stationary with respect to admissible displacements". That is,

$$\partial \Pi / \partial q_i = 0, \quad i=1,2,\dots,N \quad (2.23)$$

Furthermore, this equilibrium state is stable, if Π is a relative minimum there, and unstable, if Π is a relative maximum there.

For Π to be a minimum,

$$\partial^2 \Pi / \partial q_i^2 > 0 \quad (2.24)$$

and the equilibrium state is stable.

For Π to be a maximum,

$$\partial^2 \Pi / \partial q_i^2 < 0 \quad (2.25)$$

and the equilibrium is unstable.

For a condition of neutral equilibrium.

$$\partial^2 \Pi / \partial q_i^2 = 0. \quad (2.26)$$

For continuous systems (elastic bodies), the principle of stationary potential energy is written in variational form as:

$$\delta \Pi = 0 \quad \text{or} \quad \delta(U + V) = 0. \quad (2.27)$$

Likewise, for a condition of stable equilibrium, $\delta^2 \Pi > 0$. For a condition of instability, $\delta^2 \Pi < 0$ and for neutral equilibrium, $\delta^2 \Pi = 0$.

The use of the above principle to establish the equilibrium conditions of the system requires us to resort to the calculus of variations.

Calculus of variations deals with the evaluation of stationary values or extremals (maximum or minimum) of functionals. Functionals are definite integrals having functions as arguments. The functions in these integrals are unknowns and the calculus of variations is used to determine the conditions under which these functionals will assume a stationary value. In applying the calculus of variations to extremize a functional, one obtains conditions that the functions must satisfy to ensure that the functional will assume a stationary value. In contrast, in applying the ordinary calculus to extremize a function, one obtains the value of the independent variable for which the function will assume an extremum value. U and V should be written as functionals of the deflection (or deformation) and then, we apply the calculus of variation to make Π stationary with respect to admissible variations [16].

2.5 Dynamic Stability

In dynamic stability, there are changes in the kinetic energy of the system. This analysis is performed when we are interested in the stability of the system or beam under time-varying force or when nonconservative forces are present. The energy transformations are time dependent and Hamilton's principle must be fully implemented to derive the equations of motion. The solution to the equations of motion is accomplished by assuming an admissible response which can be an eigenfunction from an NDOF free vibration analysis or another admissible function that properly reflects the geometric constraints of the beam. The response is then substituted into the equations of motion and simplified into a system of N linear homogeneous equations whose nontrivial solutions are determined by setting the determinant of the coefficient matrix to zero. The resulting equation is called the characteristic equation for the system. The dynamic stability criterion is then applied to this equation to determine the bifurcation or critical load. The dynamic criterion requires that the roots to the characteristic equation have no positive real terms and consequently, no unbounded responses. In other words, the critical value of the loading parameter f_x is its smallest positive value which leads to multiple roots of the characteristic equation with no positive real terms.

Figure 11 illustrates a plot of the response of second order systems depending on the location of the roots of the characteristic equations and the damping factor[18]. Several methods have been applied to determine the existence of roots in the right half of the complex plane. A root of the characteristic equation may be defined by $s = \sigma \pm j\omega$, where s denotes a root in the complex plane, σ is the real part and ω is the

magnitude of the imaginary part. Figure 12 shows the root locus of a second order linear system as a function of the damping ratio ζ [18]. Notice that the roots will be on the right-hand side of the s-plane for negative damping values. These roots result in unbounded and thus unstable responses.

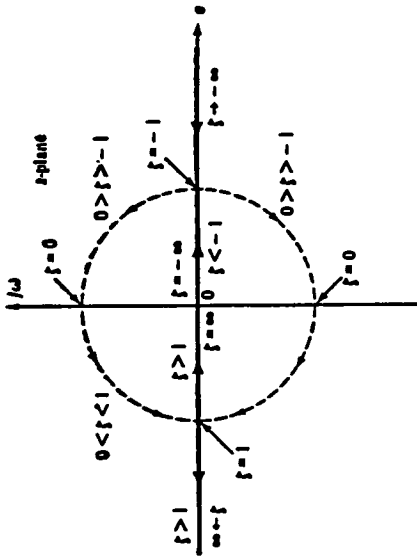


Figure 11:

Locus of roots of a second order linear system $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0$ when ω_n is constant and the damping ratio, ζ , is varied from $-\infty$ to $+\infty$.

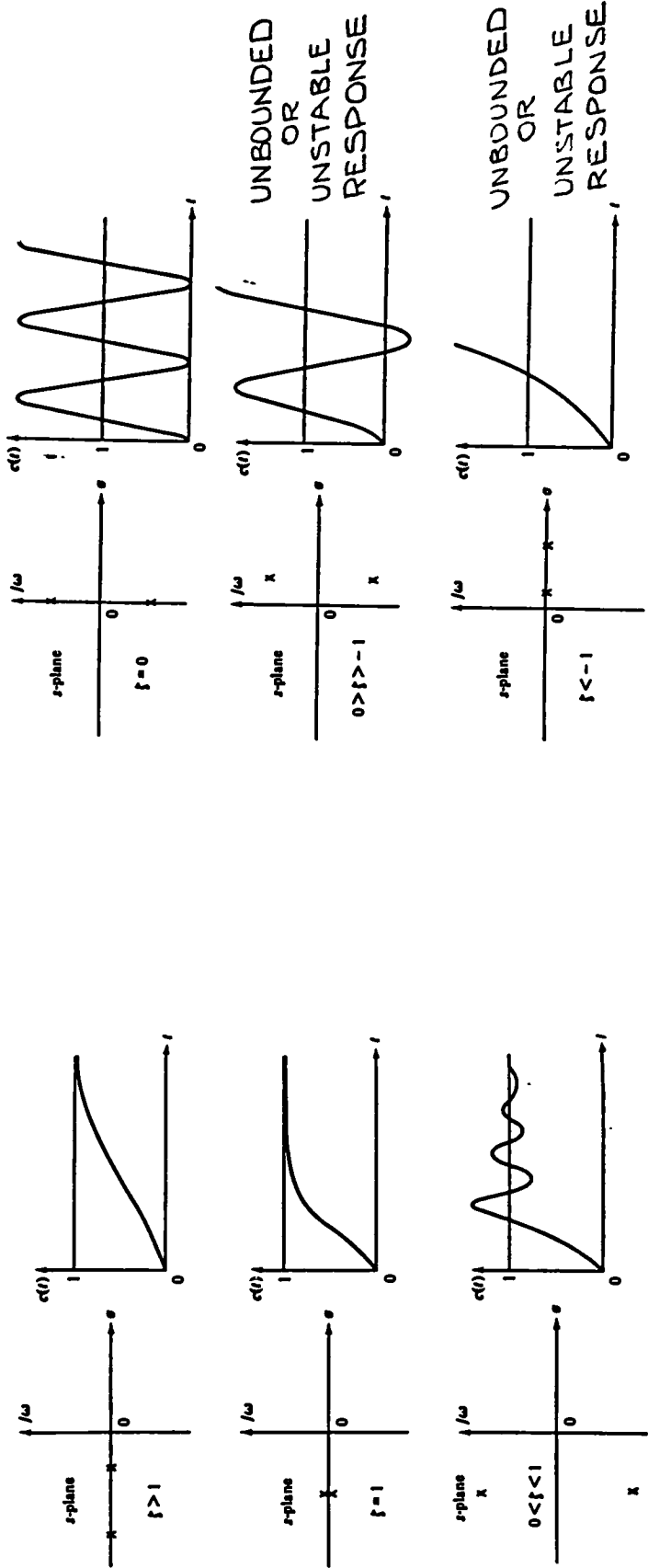


Figure 12:

Response of second order linear system for various locations of roots in the s-plane

[3] The Governing Dynamical System

3.1 Derivation of Governing Equations

The equations of motion governing the loaded cantilever beam described in section 1 are obtained via Hamilton's principle:

$$\int_{t_1}^{t_2} \delta(T - V)^* dt + \int_{t_1}^{t_2} \delta W_{nc}^* dt = 0 \quad (3-1)$$

where

T = total kinetic energy of the system

V = potential energy of the system, including the strain energy and the potential energy of conservative external forces

δW_{nc} = virtual work done by nonconservative forces, including damping forces and external forces not accounted for in V .

t_1, t_2 = times at which the configuration of the system is known.

At a point x along the neutral axis of the beam, a displacement is defined by orthogonal components, $u(x,t)$, in the axial direction and $v(x,t)$, in the transverse direction (**figure 9**).

We assume that the beam displacements, $u(x,t)$ and $v(x,t)$, can be written as the sum of deflections in the N lowest modes of a uniform cantilever elastic beam. That is:

$$u(x,t) = \sum_{i=1}^N \psi_i(x) * u_i(t) \quad (3.2-a)$$

$$v(x,t) = \sum_{i=1}^N \phi_i(x) * v_i(t) \quad (3.2-a)$$

This procedure is known as Galerkin's method, which is widely used to approximate continuous systems by an equivalent system having a finite number of degrees of freedom[19]. In the Galerkin's method, $u(x,t)$ and $v(x,t)$ are selected in such a way as to satisfy both the natural boundary conditions and the geometric boundary conditions. Another method called Raleigh-Ritz selects $u(x,t)$ and $v(x,t)$ that need to satisfy only the geometric boundary conditions [7,16].

The total kinetic energy of the system is given by:

$$T = 0.5 \cdot \rho \int_0^L [(u_t)^2 + (v_t)^2] dx \quad (3.3)$$

where ρ is the beam density per unit length.

The total elastic or strain energy due to bending and compression is given by:

$$U_T = 0.5 \int_0^L [E \cdot I \cdot (v'')^2] dx + 0.5 \int_0^L [E \cdot A \cdot (u' + 0.5 \cdot v'^2)^2] dx \quad (3.4)$$

The moment, axial and transverse component of the load are assumed to be concentrated at the free end of the beam. The potential energy attained by these loads can be described as:

$$V_T = -W = f_x \cdot u(L,t) + M \cdot v'(L,t) - f_y \cdot v(L,t) \quad (3.5)$$

where W is the total work done by the external forces and moments.

The total potential energy of the beam, V or Π , is the sum of the strain energy, U_T , and the potential energy of the external loads, V_T , that is:

$$\Pi = V = U_T + V_T = 0.5 \int_0^L [E \cdot I \cdot (v'')^2] \cdot dx + 0.5 \int_0^L [E \cdot A \cdot (u' + 0.5 \cdot v'^2)^2] \cdot dx + f_x \cdot u(L,t) + M \cdot v'(L,t) - f_y \cdot v(L,t) \quad (3.6)$$

Applying the variational operator to T and V , we get:

$$\delta T = \rho \int_0^L [(u_t) \cdot (\delta u_t) + (v_t) \cdot (\delta v_t)] \cdot dx \quad (3.7)$$

$$\delta V = \int_0^L [E \cdot I \cdot (v'') \cdot (\delta v'')] \cdot dx + \int_0^L [E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot \delta(u' + 0.5 \cdot v'^2)] \cdot dx + f_x \cdot \delta u(L,t) + M \cdot \delta v'(L,t) - f_y \cdot \delta v(L,t) \quad (3.8)$$

Integrating δV by parts twice, we have:

$$\begin{aligned} \delta V = & E \cdot I \cdot (v'') \cdot (\delta v'') \Big|_0^L - E \cdot I \cdot (v''') \cdot \delta v \Big|_0^L + \int_0^L [E \cdot I \cdot (v''''') \cdot \delta v] \cdot dx \\ & + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot \delta u \Big|_0^L - \int_0^L [E \cdot A \cdot (u' + 0.5 \cdot v'^2)' \cdot \delta u] \cdot dx \\ & + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' \cdot \delta v \Big|_0^L - \int_0^L \{ E \cdot A \cdot [(u' + 0.5 \cdot v'^2) \cdot v']' \cdot \delta v \} \cdot dx \\ & + f_x \cdot \delta u(L,t) + M \cdot \delta v'(L,t) - f_y \cdot \delta v(L,t) \end{aligned} \quad (3.9)$$

Substituting eq. (3.7) and eq. (3.9) into eq. (3.1), we get:

$$\begin{aligned}
\delta(T - V) = & \int_{t_1}^{t_2} \left\{ \rho \int_0^L [(u_t)(\delta u_t) + (v_t)(\delta v_t)] dx \right\} dt \\
& - \int_{t_1}^{t_2} \{ E I (v'')(\delta v') \Big|_0^L - E I (v''') \delta v \Big|_0^L + \int_0^L [E I (v'''') \delta v] dx \\
& + E A (u' + 0.5 v'^2) \delta u \Big|_0^L - \int_0^L [E A (u' + 0.5 v'^2)' \delta u] dx \\
& + E A (u' + 0.5 v'^2) v' \delta v \Big|_0^L - \int_0^L \{ E A [(u' + 0.5 v'^2) v']' \delta v \} dx \\
& + f_x \delta u(L, t) + M \delta v(L, t) - f_y \delta v(L, t) \} dt = 0 \quad (3.10)
\end{aligned}$$

Notice that the first term in the above expression can be integrated by parts to get:

$$\begin{aligned}
\int_{t_1}^{t_2} \delta T dt = & \int_{t_1}^{t_2} \left\{ \rho \int_0^L [u_t \delta u_t + v_t \delta v_t] dx \right\} dt = \\
& \int_0^L [\rho u_t \delta u \Big|_{t=t_2} - \rho u_t \delta u \Big|_{t=t_1}] dx - \int_{t_1}^{t_2} \int_0^L \rho u_{tt} \delta u dx dt \\
& + \int_0^L [\rho v_t \delta v \Big|_{t=t_2} - \rho v_t \delta v \Big|_{t=t_1}] dx - \int_{t_1}^{t_2} \int_0^L \rho v_{tt} \delta v dx dt \quad (3.11)
\end{aligned}$$

Substituting eq. (3.11) into eq. (3.10), we have:

$$\begin{aligned}
& \int_0^L [\rho^* u_t^* \delta u|_{t=t_2} - \rho^* u_t^* \delta u|_{t=t_1}]^* dx - \int_{t_1}^{t_2} \int_0^L \rho^* u_{tt}^* \delta u^* dx^* dt \\
& + \int_0^L [\rho^* v_t^* \delta v|_{t=t_2} - \rho^* v_t^* \delta v|_{t=t_1}]^* dx - \int_{t_1}^{t_2} \int_0^L \rho^* v_{tt}^* \delta v^* dx^* dt \\
& - \int_{t_1}^{t_2} \{ E^* I^* (v'')^* \delta v|_0^L - E^* I^* (v''')^* \delta v|_0^L + \int_0^L [E^* I^* (v''')^* \delta v]^* dx \\
& + E^* A^* (u' + 0.5^* v'^2)^* \delta u|_0^L - \int_0^L [E^* A^* (u' + 0.5^* v'^2)^* \delta u]^* dx \\
& + E^* A^* (u' + 0.5^* v'^2)^* v'^* \delta v|_0^L - \int_0^L \{ E^* A^* [(u' + 0.5^* v'^2)^* v']^* \delta v \}^* dx \\
& + f_x^* \delta u(L, t) + M^* \delta v'(L, t) - f_y^* \delta v(L, t) \}^* dt = 0 \tag{3.12}
\end{aligned}$$

Grouping like terms, we get:

$$\begin{aligned}
& \int_0^L [\rho^* u_t^* \delta u|_{t=t_2} - \rho^* u_t^* \delta u|_{t=t_1}]^* dx + \int_0^L [\rho^* v_t^* \delta v|_{t=t_2} - \rho^* v_t^* \delta v|_{t=t_1}]^* dx \\
& - \int_{t_1}^{t_2} \int_0^L \{ \rho^* u_{tt} - E^* A^* (u' + 0.5^* v'^2)' \}^* \delta u^* dx^* dt \\
& - \int_{t_1}^{t_2} \int_0^L \{ \rho^* v_{tt} + E^* I^* v'''' - E^* A^* [(u' + 0.5^* v'^2)^* v']^* \}^* \delta v^* dx^* dt
\end{aligned}$$

$$\begin{aligned}
& - \int_{t_1}^{t_2} \{ E \cdot I \cdot v'' + M \} \cdot \delta v'|_{x=L} \cdot dt - \int_{t_1}^{t_2} E \cdot I \cdot v'' \cdot \delta v'|_{x=0} \cdot dt \\
& - \int_{t_1}^{t_2} \{ - E \cdot I \cdot v''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' - f_y \} \cdot \delta v|_{x=L} \cdot dt \\
& - \int_{t_1}^{t_2} \{ - E \cdot I \cdot v''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' \} \cdot \delta v|_{x=0} \cdot dt \\
& - \int_{t_1}^{t_2} \{ E \cdot A \cdot (u' + 0.5 \cdot v'^2) + f_x \} \cdot \delta u|_{x=L} \cdot dt - \int_{t_1}^{t_2} \{ E \cdot A \cdot (u' + 0.5 \cdot v'^2) \} \cdot \delta u|_{x=0} \cdot dt = 0 \quad (3.13)
\end{aligned}$$

For the above equation to be satisfied, the following conditions must be held:

$$\rho \cdot u_1 \cdot \delta u|_{t=t_2} = 0 \quad (3.13-i)$$

$$\rho \cdot u_1 \cdot \delta u|_{t=t_1} = 0 \quad (3.13-ii)$$

$$\rho \cdot v_1 \cdot \delta v|_{t=t_2} = 0 \quad (3.13-iii)$$

$$\rho \cdot v_1 \cdot \delta v|_{t=t_1} = 0 \quad (3.13-iv)$$

$$\{ \rho \cdot u_{tt} - E \cdot A \cdot (u' + 0.5 \cdot v'^2)' \} \cdot \delta u = 0 \quad (3.13-v)$$

$$\{ \rho \cdot v_{tt} + E \cdot I \cdot v'''' - E \cdot A \cdot [(u' + 0.5 \cdot v'^2) \cdot v]' \} \cdot \delta v = 0 \quad (3.13-vi)$$

$$\{ E \cdot I \cdot v'' + M \} \cdot \delta v'|_{x=L} = 0 \quad (3.13-vii)$$

$$E \cdot I \cdot v'' \cdot \delta v' |_{x=0} = 0 \quad (3.13-viii)$$

$$\{- E \cdot I \cdot v''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' - f_y \} \cdot \delta v |_{x=L} = 0 \quad (3.13-ix)$$

$$\{- E \cdot I \cdot v''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' \} \cdot \delta v |_{x=0} = 0 \quad (3.13-x)$$

$$\{ E \cdot A \cdot (u' + 0.5 \cdot v'^2) + f_x \} \cdot \delta u |_{x=L} = 0 \quad (3.13-xi)$$

$$E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot \delta u |_{x=0} = 0 \quad (3.13-xii)$$

In using Hamilton's principle, it is assumed that the configuration is specified at times t_1 and t_2 . For the cantilever beam, this means that

$$\delta u(x, t_1) = \delta u(x, t_2) = \delta v(x, t_1) = \delta v(x, t_2) = 0 \quad (3.14)$$

which takes care of the first four conditions above[13].

Since δu , δv are not arbitrarily equal to zero, it follows from eq. (3.13-v) and (3.13-vi) that:

$$\rho \cdot u_{tt} - E \cdot A \cdot (u' + 0.5 \cdot v'^2)' = 0 \quad (3.15)$$

and

$$\rho \cdot v_{tt} + E \cdot I \cdot v'''' - E \cdot A \cdot [(u' + 0.5 \cdot v'^2) \cdot v']' = 0 \quad (3.16)$$

which are the equations of motion for axial and transverse vibrations of the beam.

The other conditions are called natural and geometric boundary conditions for the cantilever beam.

At the fixed end ($x=0$), δu and δv are zero, therefore from equations {3.13-xii,-x):

$$u(0,t) = 0 \quad (3.17)$$

and

$$v(0,t) = 0. \quad (3.18)$$

At the free end ($x=L$), δu and δv are not necessarily zero, hence from equations (3.13-xi,-ix):

$$E \cdot A \cdot (u' + 0.5 \cdot v'^2) + f_x = 0$$

$$\text{or } f_x = - E \cdot A \cdot (u' + 0.5 \cdot v'^2) \quad (3.19)$$

$$\text{and } - E \cdot I \cdot v''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' - f_y = 0 \quad (3.20)$$

Finally, at the fixed end the slope $\delta v'$ is zero and at the free end, the slope is not necessarily equal to zero. Therefore from equations (3.13-vii, -viii)

$$v'(0,t) = 0 \quad (3.21)$$

and

$$E \cdot I \cdot v'' + M = 0 \quad \text{or} \quad v'' = -M/(E \cdot I) \quad \text{at } x=L \quad (3.22)$$

We may linearize the equations of motion by striking out the nonlinear term $0.5 \cdot v'^2$ to arrive at a linear differential equation with linear boundary conditions for the axial displacement $u(x,t)$. The resulting

longitudinal motion is generally called the primary motion and it is calculated on a linearized basis accordingly to the usual procedure in stability theory. A solution for the primary motion is applied to the transverse equation of motion which will lead to an equation for $v(x,t)$ alone. This resulting transverse motion is called the secondary motion and it is represented by a linear differential equation with either constant coefficients or time-dependent coefficients[4].

The elimination of $0.5 \cdot v'^2$ will lead to the following set of equations:

$$\rho \cdot u_{tt} - E \cdot A \cdot u'' = 0 \quad (3.15a)$$

$$\rho \cdot v_{tt} + E \cdot I \cdot v'''' - E \cdot A \cdot [u' \cdot v']' = 0 \quad (3.16a)$$

$$\text{or } \rho \cdot v_{tt} + E \cdot I \cdot v'''' - E \cdot A \cdot [u' \cdot v'' + u'' \cdot v'] = 0 \quad (3.16b)$$

$$u(0,t) = 0 \quad (3.17)$$

$$v(0,t) = 0 \quad (3.18)$$

$$f_x = - E \cdot A \cdot u'(L,t) \quad (3.19a)$$

$$- E \cdot I \cdot v'''' + E \cdot A \cdot u' \cdot v' - f_y = 0 \text{ at } x=L \quad (3.20a)$$

$$\text{or } E \cdot I \cdot v''''(L,t) + f_x \cdot v'(L,t) + f_y = 0 \quad (3.20b)$$

$$v'(0,t) = 0 \quad (3.21)$$

$$E \cdot I \cdot v'' + M = 0 \quad \text{or} \quad v'' = -M/(E \cdot I) \quad \text{at } x=L \quad (3.22)$$

The above set of equations can be divided into two sets: one which is dependent on $u(x,t)$ only and the other set which is dependent on both $u(x,t)$ and $v(x,t)$.

The first set is based on the following equations:

$$\rho \cdot u_{tt} - E \cdot A \cdot u'' = 0 \quad (3.15a)$$

$$u(0,t) = 0 \quad (3.17)$$

$$f_x = - E \cdot A \cdot u'(L,t) \quad (3.19a)$$

Once a solution is found for this set, one may substitute the result into the second set to make it dependent on only $v(x,t)$.

The second set is based on the following equations:

$$\rho * v_{tt} + E * I * v'''' - E * A * [u' * v'' + u'' * v'] = 0 \quad (3.16b)$$

$$v(0,t) = 0 \quad (3.18)$$

$$E * I * v'''(L,t) + f_x * v'(L,t) + f_y = 0 \quad (3.20b)$$

$$v'(0,t) = 0 \quad (3.21)$$

$$v''(L,t) = -M/(E * I) \quad (3.22)$$

To simplify the problem, we may assume the load f_x to be either constant or a slowly varying function of time (figure 13) such that:

$$f_x = f_{x0} + f_{x1} * \cos(\omega * t) \text{ with constants } f_{x0}, f_{x1} \quad (3.23)$$

and with

$$f_{x0} \gg f_{x1} \quad (3.24)$$

and that

$$\omega \ll \text{fundamental frequency of longitudinal vibration}[4]. \quad (3.25)$$

For such a case, one may consider f_x as approximately a constant and use this approximation to solve the first set of equations. The resulting solution is then:

$$u(x,t) = - f_x * x / (E * A) \quad (3.26)$$

since it satisfies all three equations in the first set. Notice also that the following conditions are then true:

$$u'(x,t) = - f_x / (E * A) \quad (3.27)$$

$$u''(x,t) = 0 \quad (3.28)$$

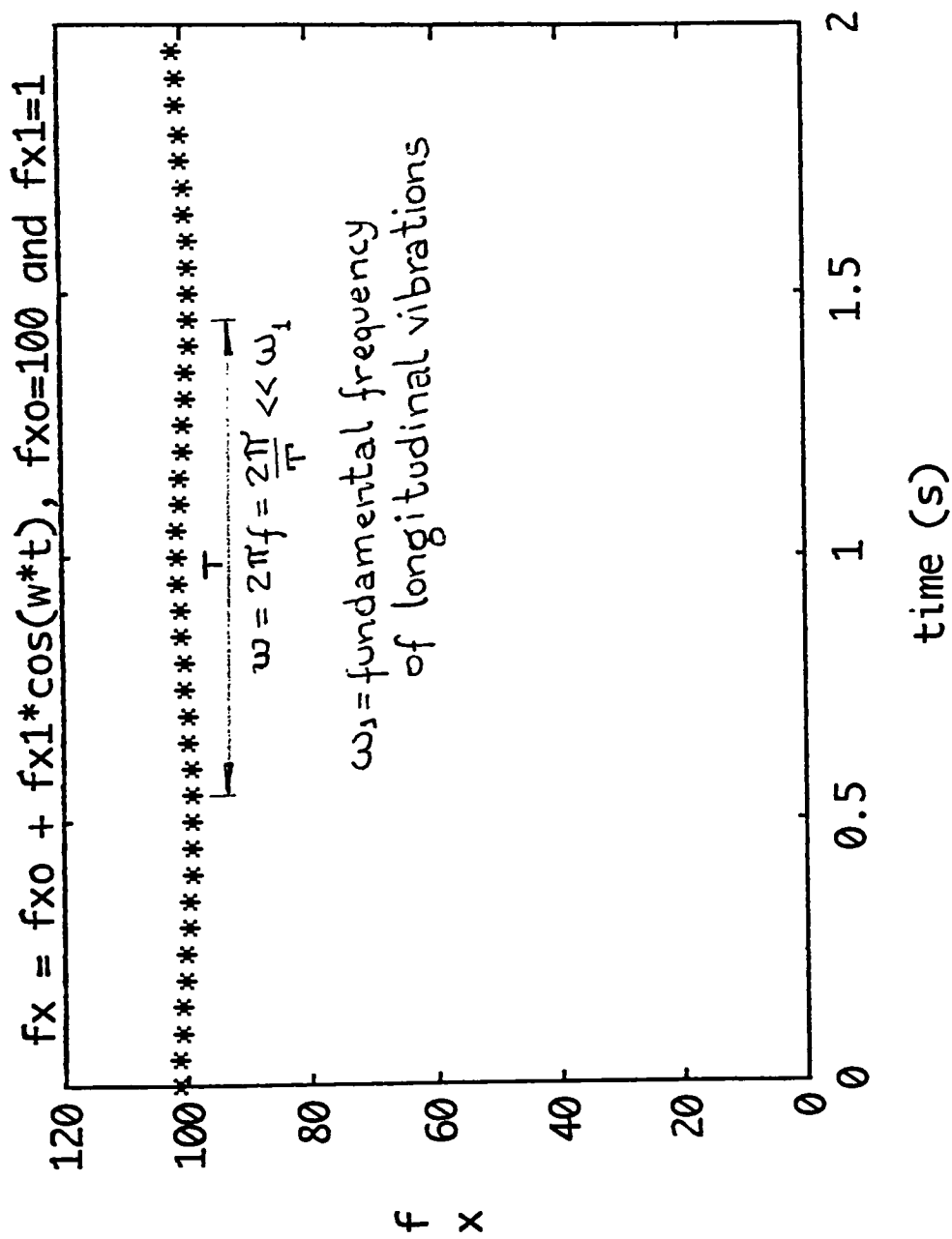


FIGURE 13

and

$$u_{tt}(x,t) = 0 \quad (3.29)$$

We can now substitute the above results into the second set of equations to uncouple it from $u(x,t)$ variable. The resulting equations will then be:

$$\rho * v_{tt} + E * I * v'''' + f_x * v'' = 0 \quad (3.30)$$

$$v(0,t) = 0 \quad (3.18)$$

$$E * I * v'''(L,t) + f_x * v'(L,t) + f_y = 0 \quad (3.20b)$$

$$v'(0,t) = 0 \quad (3.21)$$

$$v''(L,t) = -M/(E * I) \quad (3.22)$$

This set of equations are now uncoupled but equation (3.30) is not integrable in its final form. We have to recourse to Galerkin variational method[4,6,7] to solve for equation (3.30). Assuming that the beam transverse deflection can be defined as:

$$v(x,t) = \phi(x) * v(t), \quad (3.31)$$

where $\phi(x)$ is an admissible function whose shape closely approximate the deformed shape of a cantilevered beam and $v(t)$ is a function of time.

The admissible function, $\phi(x)$, could be defined as any one of the eigenfunctions for the free vibration of a cantilevered beam or a linear combination of them. The function $\phi(x)$ can also represent the mode shapes associated with the beam mode shape frequency ω [6,7,16,20].

Levinson [7] showed a solution to a similar problem using Galerkin's method and chose as a trial function:

$$\phi(x) = a_1 \cdot \phi_1(x) + a_2 \cdot \phi_2(x) \quad (3.32)$$

where a_1, a_2 are constants
and $\phi_1(x)$ and $\phi_2(x)$ are the first two eigenfunctions of the free vibration problem which are derived in the Appendix.

Another set of admissible functions often used comes in the form of a polynomial whose shape closely resembles that of the deformed shape of the beam. As an example of admissible functions for a cantilever beam, we have[16]:

$$\phi(x) = k \cdot [1 - \cos(0.5 \cdot \pi \cdot x/L)], \quad k = \text{constant} \quad (3.33)$$

Substituting equation (3.31) into the set of governing equations for the transverse motion, we get:

$$\rho \cdot v_{tt} \cdot \phi + E \cdot I \cdot v \cdot \phi'''' + f_x \cdot v \cdot \phi'' = 0 \quad (3.30a)$$

$$\phi(0) = 0 \quad (3.18a)$$

$$E \cdot I \cdot v \cdot \phi'''(L) + f_x \cdot v \cdot \phi'(L) + f_y = 0 \quad (3.20c)$$

$$\phi'(0) = 0 \quad (3.21a)$$

$$v \cdot \phi''(L) = -M/(E \cdot I) \quad (3.22a)$$

In the last equation, if we were to neglect the effect of the moment at the end of the beam, then we would arrive at the final form:

$$\phi''(L) = 0 \quad (3.22b)$$

Since the integral over the length of the beam of equation (3.30a) is not equal to zero due to the approximation used for $v(x,t)$, we need to apply Galerkin's method. In this method, we would multiply the integrand by the admissible function $\phi(x)$ and require the integrand and $\phi(x)$ to be orthogonal. This resulting integration would then be equal to zero and its computation would result in a differential equation in time only (or a set of differential equations in time only that can be treated as an eigenvalue problem to be solved). This procedure is illustrated below by the Galerkin equation:

$$\int_0^L [\rho v_{tt} \phi + E I v \phi'''' + f_x v \phi''] \phi dx = 0 \quad (3.34)$$

If we were to use Levinson's admissible function, we would arrive at two Galerkin equations:

$$\int_0^L [\rho v_{tt} \phi + E I v \phi'''' + f_x v \phi''] \phi_1 dx = 0 \quad (3.35)$$

and

$$\int_0^L [\rho v_{tt} \phi + E I v \phi'''' + f_x v \phi''] \phi_2 dx = 0 \quad (3.36)$$

which are solved to find a set of time differential equations that need to be solved as an eigenvalue problem. A treatment of the problem will be presented dealing with only the first eigenfunction.

3.3 Modal Analysis of A Cantilevered Beam

In Appendix B, the equations governing the free vibrations of a cantilever beam in both transverse and longitudinal directions are derived. The derivations yield an eigenvalue problem that is solved to arrive at the eigenfunctions for the cantilever beam. The general form of the eigenfunctions for the cantilever beam is given by [2,13]:

$$\phi_i(x) = \cosh \lambda_i x - \cos \lambda_i x - \alpha_i (\sinh \lambda_i x - \sin \lambda_i x) \quad (3.37)$$

where

$$\omega_i = \lambda_i^2 (E^* I / \rho)^{0.5} \quad (3.38)$$

and

$$\alpha_i = (\cos \lambda_i L + \cosh \lambda_i L) / (\sin \lambda_i L + \sinh \lambda_i L), \quad i=1,2,3,\dots,N \quad (3.39)$$

The terms λ_i are the constants appearing in the natural frequencies (ω_i 's) of a uniform cantilever beam in free vibration. The terms α_i are the constants in the characteristic mode shapes, $\phi_i(x)$, of cantilevered beam vibration.

The modal functions are normalized so that:

$$\int_0^L [\phi_i(x)]^2 dx = L \quad (i = 1, 2, 3, \dots, N) \quad (3.40)$$

The values of the above constants are computed numerically as:

$$\begin{aligned} \alpha_1 &= 0.7340955 & \lambda_1 * L &= 1.8751, & \alpha_2 &= 1.0184673 & \lambda_2 * L &= 4.6941, \\ \alpha_3 &= 0.9992244 & \lambda_3 * L &= 7.8548, \text{ etc.} \end{aligned}$$

The first two eigenfunctions for the transverse vibrations may be written as:

$$\begin{aligned}\phi_1(x) = & 1.0 * (\cosh 1.8751x/L - \cos 1.8751x/L) \\ & - 0.7340955 * (\sinh 1.8751x/L - \sin 1.8751x/L) \quad (3.41)\end{aligned}$$

$$\begin{aligned}\phi_2(x) = & 1.0 * (\cosh 4.6941x/L - \cos 4.6941x/L) \\ & - 1.0184673 * (\sinh 4.6941x/L - \sin 4.6941x/L) \quad (3.42)\end{aligned}$$

3.4 Numerical Values of Parameter

Numerical values for the parameters and the physical constants used in numerical computations and numerical simulations are those stated in the definition of the problem along with typical geometries used in practical applications. These physical constants are listed in a table at Appendix C.

3.5 Governing Equations

The governing time differential equations are arrived at via a MAPLE [21] program called GALERK2. which computes the Galerkin equations for a given admissible shape function(s) or eigenfunction(s). The program reads as input the number of degrees of freedom, n , and the corresponding eigenfunctions, ϕ_i , derived in section 3.3. The program then computes the Galerkin equations. If more than one eigenfunction is given, the program computes entries for a coefficient matrix B whose entries are $b_{i,j}$. These entries can be described as the coefficients of factors a_1 and a_2 in Levinson's approximation for $\phi(x)$, for instance. The resulting system of time differential equations can be described as the following eigenvalue problem:

$$[B] \cdot [a_i] = 0. \quad (3.43)$$

The determinant of the coefficient matrix is set to zero to find the characteristic equation for the system. The roots of the characteristic equation must be determined to satisfy the dynamic criterion of stability (similar to Root Locus and Stability analysis of Control System theory).

In this paper, we will utilize only the first eigenfunction, $\phi_1(x)$. The resulting computation from GALERK2 is only one time differential equation that will be examined for its stability. Consequently, we must enter the following input data into Maple at each command line:

```
> n:=1; (indicates only the first eigenfunction will be used)
> read `y12`; (read a Maple file called y12. which has the eigenfunctions)
> read `galerk2`; (read a Maple file called galerk2. which calculated the
Galerkin integral equation(s) or the entries of the coefficient matrix B)
```

The output is the following time differential equation:

$$12.35962 \cdot (E \cdot I / L^3) \cdot v + 0.871558 \cdot (f_x / L) \cdot v + \rho \cdot L \cdot v_{tt} = 0 \quad (3.44)$$

This equation can be rewritten as:

$$v_{tt} + \left[12.35962 \cdot E \cdot I / (\rho \cdot L^4) + 0.871558 \cdot f_x / (\rho \cdot L^2) \right] \cdot v = 0 \quad (3.45)$$

but

$$f_x = f_{x0} + f_{x1} \cdot \cos(\omega \cdot t) \quad (3.46)$$

Therefore,

$$v_{tt} + \left[12.35962 \cdot E \cdot I / (\rho \cdot L^4) + 0.871558 \cdot f_{x0} / (\rho \cdot L^2) + 0.871558 \cdot f_{x1} \cdot \cos(\omega \cdot t) / (\rho \cdot L^2) \right] \cdot v = 0 \quad (3.47)$$

Notice that the first coefficient of $v(t)$ is the same as the first natural frequency, ω_1 , for the free vibration of the beam.

The above equation can be expressed in the following form:

$$v_{tt} + [a + b \cos(\omega t)] * v = 0 \quad (3.48)$$

which is known as Mathieu's equation, a time differential equation with a periodic function as the coefficient of the first order term . Solutions to Mathieu's equations is widely known in the literature [4,5,6,8,9,10,11,12, 15,22]. The solutions will be stable depending on the location of the roots of the characteristic equation or the values of the coefficients a, b and the frequency ω .

Mathieu's equation can also be written into another form by replacing t, the independent variable, for a new variable τ such that[14]:

$$\tau = \omega t. \quad (3.49)$$

Since v_{tt} equals $\omega^2 v_{\tau\tau}$, the equation can be written as:

$$v_{\tau\tau}(\tau) + (a_1 + b_1 \cos \tau) * v(\tau) = 0 \quad (3.50)$$

$$\text{where} \quad a_1 = a / \omega^2 \quad (3.51)$$

$$b_1 = b / \omega^2 \quad (3.52)$$

The character of the solution of the above equation depends on the values of a_1 and b_1 . For some values of a_1 and b_1 , the response will grow with time resulting in unstable behavior. A stability plot for the above equation is known as a Strutt's diagram [15] which is illustrated in **figure 14**. Shaded areas in this plot show regions where the beam response will be stable. The axes are represented by the coefficients a_1 and b_1 . We will explore the stability of such an equation in the next section.

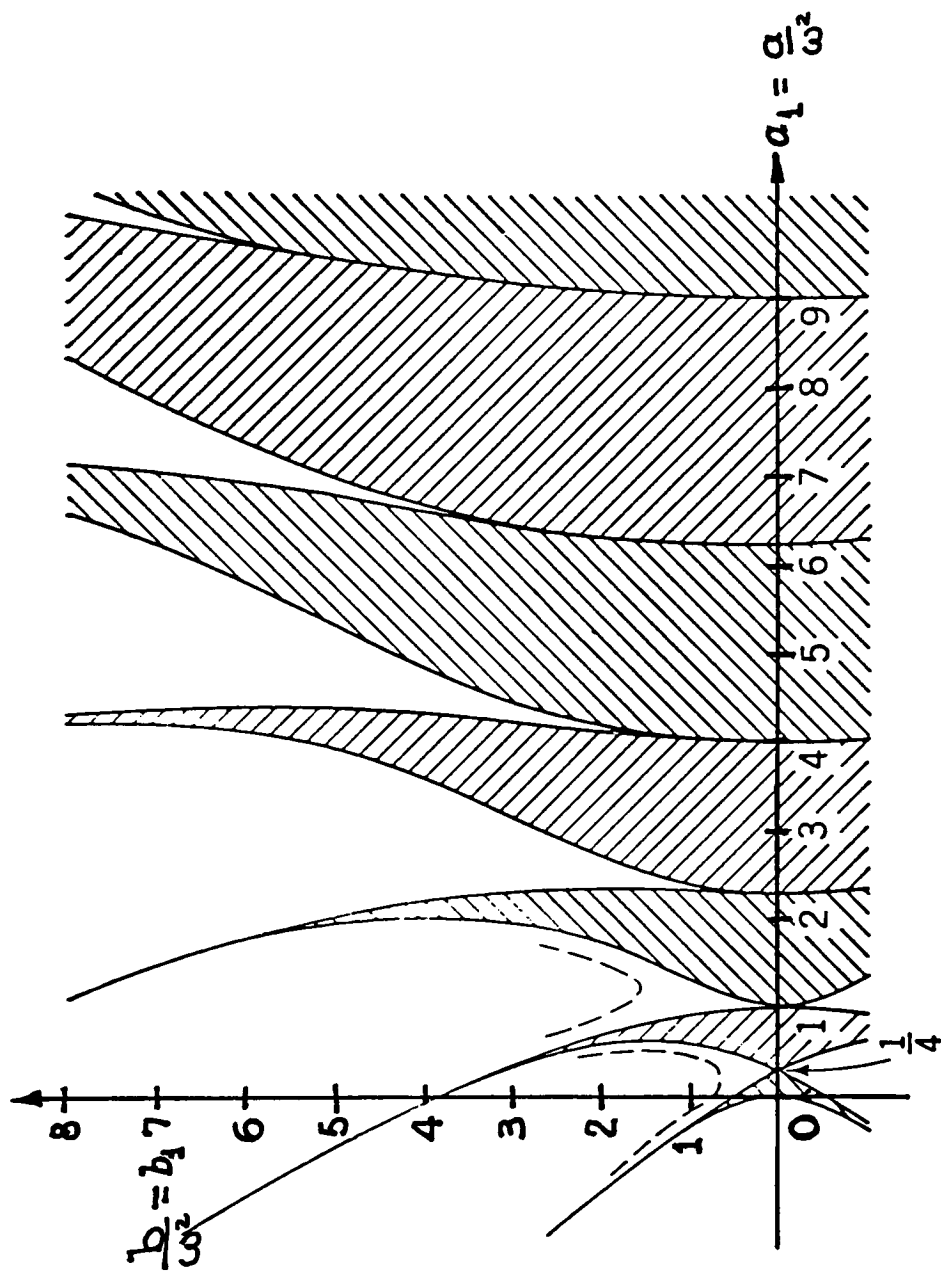


Figure 14 : STRUTT'S DIAGRAM

Stability plot for Mathieu's equation in the form: $v_{\tau\tau} + (a_1 + b_1 \cos(\tau))v = 0$.
 Shade regions represent stable responses. Solid lines represent the transition from stable to unstable behavior. Dashed lines represent the effect of damping on the system.

[4] Dynamical Stability Analysis

4.1 Bifurcation and Eigenvalue Approaches

In the bifurcation approach, one determines the conditions upon which there is a change in quality of the behavior of the system configuration. For a second order linear system such as mass supported by a spring and a dashpot, the characteristic equation can be represented by a quadratic equation, $a*s^2 + b*s + c = 0$. The bifurcation represents the conditions whereby the roots, s , of the equation change in quality from real to complex roots. In the complex plane, for instance, roots which have imaginary parts result in oscillatory motion. The response may be decaying with time (real parts are negative) or increasing with time (real parts are positive) depending on the location of the real parts of the roots. When the real parts are zeros, the resulting motion is undamped oscillation. Stable motion results when the real parts of the roots are negative values so that the response is always bounded (figures 11-12).

In the cases of differential equations with periodic coefficients such as in Mathieu's equation, one applies a method developed by Floquet to derive the stability conditions [6, 8-12, 23].

Using Floquet's method, one writes the differential equation in vector form (state space in control theory), for instance:

$$\text{Given:} \quad v_{\tau\tau} + (a_1 + b_1*\cos(\tau))*v(\tau) = 0 \quad (4.1)$$

We can write that

$$v_{1\tau} = v_2 \quad (4.2)$$

and

$$v_{2\tau} = - (a_1 + b_1*\cos(\tau))*v_1. \quad (4.3)$$

In matrix form:

$$\begin{bmatrix} v_{1\tau} \\ v_{2\tau} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(a + b \cos(\tau)) & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (4.4)$$

Also:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_{1\tau} \end{bmatrix} \quad (4.5)$$

Due to the fact that Mathieu's equation is a linear and homogeneous differential equation with coefficients which are periodic functions of τ , it follows that there exists a pair of linearly independent and periodic solutions $w_1(\tau)$ and $w_2(\tau)$ other than nonzero or trivial solutions and every other solution $w(\tau)$ is a linear combination of w_1 and w_2 such as:

$$w(\tau) = k_1 w_1 + k_2 w_2. \quad (4.6)$$

This pair of solutions w_1, w_2 is called a fundamental set of solutions.

The requirement for w_1 and w_2 to form a fundamental set is that the Wronskian determinant should not vanish in τ . Let $|W|$ be the Wronskian determinant, then:

$$|W| = \begin{vmatrix} w_1 & w_2 \\ w_{1\tau} & w_{2\tau} \end{vmatrix} \quad (4.7)$$

It is noted that if $|W| = 0$ for any value of τ then it vanishes for all other values of τ .

Furthermore, if $w_1(\tau)$ and $w_2(\tau)$ form a fundamental set of solutions, it follows that $w_1(\tau + 2\pi)$ and $w_2(\tau + 2\pi)$ also form a fundamental set since they also satisfy Mathieu's equation. This is due to the fact that the coefficient $\cos(\tau) = \cos(\tau + 2\pi)$ and $|W(\tau + 2\pi)|$ does not vanish.

We can also write the following equations based on points presented above:

$$w_1(\tau + 2\pi) = a_{11} * w_1(\tau) + a_{12} * w_2(\tau) \quad (4.8)$$

$$\text{and } w_2(\tau + 2\pi) = a_{21} * w_1(\tau) + a_{22} * w_2(\tau) \quad (4.9)$$

The Wronskian of the above terms can be represented as:

$$| W(\tau + 2\pi) | = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = | W(\tau) | \quad (4.10)$$

Since the Wronskian cannot be zero at any value of τ , it follows that

$$| A | = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0 \quad (4.11)$$

There are also solutions to Mathieu's equation that have the additional property that when τ is shifted by the period 2π , their value is multiplied by a constant. This can be written as:

$$w(\tau + 2\pi) = \sigma * w(\tau) \quad \text{with } \sigma, \text{ a constant.} \quad (4.12)$$

These solutions are called normal solutions and they play a central role in Floquet theory.

Now any normal solution w can be written as a linear combination of w_1 and w_2 like:

$$w = \lambda_1 * w_1 + \lambda_2 * w_2. \quad (4.13)$$

A normal solution w satisfies equation (4.12) and w_1 and w_2 satisfy equations (4.8-9); it follows that the relation:

$$[\lambda_1(a_{11} - \sigma) + \lambda_2 a_{21}]w_1 + [\lambda_1 a_{12} + \lambda_2(a_{22} - \sigma)]w_2 = 0 \quad (4.14)$$

is valid for all values of τ , and the coefficients of w_1 and w_2 must be equal to zero. Additionally, λ_1 and λ_2 are not both equal to zero, it then follows that the characteristic equation can be written as:

$$\begin{vmatrix} a_{11} - \sigma & a_{21} \\ a_{12} & a_{22} - \sigma \end{vmatrix} = 0 \quad (4.15)$$

or

$$\sigma^2 - (a_{11} + a_{22})\sigma + (a_{11}a_{22} - a_{21}a_{12}) = 0 \quad (4.16)$$

The characteristic equation is a quadratic equation for σ whose roots do not vanish because the constant term is equal to $|A|$ which is not equal to zero. This determinant, $|A|$, represents the product of the two roots of the characteristic equation.

If the roots σ_1 and σ_2 are not equal, there exists a pair of linearly independent normal solutions. The two normal solutions consist of the product of an exponential function and a periodic function of period 2π . The general solution may be represented by:

$$w(\tau) = K_1 e^{\alpha_1 \tau} \phi_1(\tau) + K_2 e^{\alpha_2 \tau} \phi_2(\tau) \quad (4.17)$$

$$\text{where } w_1(\tau) = e^{\alpha_1 \tau} \phi_1(\tau) \quad (4.18)$$

$$w_2(\tau) = e^{\alpha_2 \tau} \phi_2(\tau) \quad (4.19)$$

$$\text{and } e^{\alpha_i 2\pi} = \sigma_i. \quad (4.20)$$

In the above terms, $\alpha_1 \neq \alpha_2$ and ϕ_1 and ϕ_2 have period 2π . Both roots and both α 's may be complex numbers.

4.2 Solutions of Characteristic Equation

The Floquet theory presented above shows some important characteristics of the solutions to Mathieu's equation which will be further developed here with the objective of deciding on the criteria for dynamic stability of these solutions. The solutions will be stable if they are bounded in time and unstable if an unbounded solution exists.

If we select as fundamental solutions w_1 and w_2 that satisfy the following initial conditions:

$$w_1(0) = 1, \quad w_{1\tau}(0) = 0 \quad (4.21)$$

$$w_2(0) = 0, \quad w_{2\tau}(0) = 1 \quad (4.22)$$

then the Wronskian determinant will be :

$$|A| = 1 \quad (4.23)$$

and this means that

$$\sigma_1^* \sigma_2 = 1 \quad (4.24)$$

Applying the above result to the characteristic equation yields:

$$\sigma^2 - S^* \sigma + 1 = 0 \quad (4.25)$$

where

$$S = \sigma_1 + \sigma_2 = a_{11} + a_{22} \quad (4.26)$$

The roots to the characteristic equation can be written as:

$$\sigma_{1,2} = [S/2] \pm [(S/2)^2 - 1]^{0.5} \quad (4.27)$$

In the case of repeated roots:

$$\sigma_1 = \sigma_2 = \sigma = 1 \quad (4.28a)$$

$$\text{or } \sigma_1 = \sigma_2 = \sigma = -1 \quad (4.28b)$$

and Mathieu's equation has a periodic solution.

For stability it is necessary to require that both $|\sigma_1|$ and $|\sigma_2|$ satisfy the inequality $|\sigma| \leq 1$, and this means that σ_1 and σ_2 must satisfy the requirement:

$$|\sigma_1| = |\sigma_2| = 1 \quad (4.29)$$

The above expressions are necessary conditions for stability in the present case. They are also sufficient conditions for stability in case $\sigma_1 \neq \sigma_2$.

In view of the stability conditions set by equation (4.29) and the fact that S is real, we can state that[14]:

(i) if $|S| > 2$, the solutions are unstable (at least one root will lie in the positive real axis of the complex plane) (4.30)

(ii) if $|S| < 2$, the solutions are stable (no root will lie in the right half of the complex plane). (4.31)

This is due to $\sigma_1 \neq \sigma_2$, $|\sigma_1| = |\sigma_2| = 1$ and α_i in equations (4.18-19) are therefore pure imaginary.

Since S is real, the transition from stable to unstable behavior occurs for $S = +2$ or $S = -2$, which corresponds to the repeated roots $\sigma = +1$ or $\sigma = -1$. At these values, the responses are sometimes called neutrally stable. (4.32)

4.3 Dynamic Stability Criterion

The dynamic criterion of stability requires that the roots of the characteristic equation for the problem be located on the imaginary axis or to the left of this axis in the complex plane. If we call these roots, σ_1 and σ_2 , the absolute value of their sum should be less than 2. If we call

their sum, S , then we can write that stable solutions will occur when:

$$|S| = |\sigma_1 + \sigma_2| \leq 2 \quad \text{or} \quad -2 \leq S \leq 2. \quad (4.33)$$

Dr. Joseph Torok has provided me with a set of MATLAB [24] programs that calculates the S values given a range for the parameters a_1 and b_1 . We need to provide the following data:

| | |
|-------------------|---|
| N | number of points that each parameter range will be subdivided into to form a grid |
| amin, amax | the minimum and maximum values for the parameter a_1 |
| bmin, bmax | the minimum and maximum values for the parameter b_1 |

We would then call MATLAB program POINTS.M to define the grid points.

After the above step, we call MATLAB program RUNPTS.M to calculate the S values for Mathieu's equation (which is a special case of Hill's equation) using ode45 solver for a time differential equation using a Runge-Kutta method.

These above program may take over one hour for $N=50$ (about 30 minutes for $N=25$) for small values of a_1 and b_1 . For the values investigated in the parametric evaluation, the grid was made coarse ($N=5$) since it took about an hour for the highest values of a_1 .

To see the values of S calculated above, just type S at the command line. Another MATLAB program called RUNPLT.M can be invoked to generate a plot of the stability surface $S(a_1, b_1)$. In order to involve the graphics display, one needs to type "terminal" at the command line and select the proper terminal. The VT240 graphics terminal was initially selected for

display but the printouts were of poor quality and the plots were redone using a Macintosh Quadra 660AV and MATLAB version 4.1. It is recommended that the data be saved after each run for later printing since it takes a long time to generate the data.

The Strutt's diagram previously presented constitutes a contour plot of the stability surface sliced at the S values equal to -2 or 2 by planes parallel to the $a_1 \cdot b_1$ plane. The solid lines are defined by $S=-2$ and $S=2$ and the shaded regions of stability represent S values within the above S values. A MATLAB program called CON.M was also created to generate the contour plots if needed.

4.4 Parametric Evaluation of Stability

From part 1, we had expressed the loads as:

$$f_y = N \cdot \sin \zeta - f \cdot \sin(90^\circ - \zeta) = N \cdot (\sin \zeta - \mu_s \cdot \cos \zeta),$$

$$f_x = N \cdot \cos \zeta + f \cdot \cos(90^\circ - \zeta) = N \cdot (\cos \zeta + \mu_s \cdot \sin \zeta)$$

$$M = f_x \cdot h/2 = (N \cdot h/2) \cdot (\cos \zeta + \mu_s \cdot \sin \zeta)$$

for the analysis, we will neglect the effect of this moment.

Here we will set $\mu_s = \mu_o + \mu_1 \cdot \cos(\omega \cdot t)$. Hence, we can write:

$$\begin{aligned} f_x &= N \cdot \cos \zeta + [\mu_o + \mu_1 \cdot \cos(\omega \cdot t)] \cdot N \cdot \sin \zeta = \\ &= N \cdot (\cos \zeta + \mu_o \cdot \sin \zeta) + \mu_1 \cdot N \cdot \sin \zeta \cdot \cos(\omega \cdot t) \end{aligned} \quad (4.34)$$

from the above expression, we can deduce that:

$$f_{x0} = N \cdot (\cos \zeta + \mu_o \cdot \sin \zeta) \quad (4.35)$$

and

$$f_{x1} = \mu_1 \cdot N \cdot \sin \zeta \quad (4.36)$$

The angle that the beam makes with the tangent to the roller at the contact point is:

$$\zeta = 90^\circ - \beta/2 - \gamma/2 - \theta + \gamma = 90^\circ - \beta/2 + \gamma/2 - \theta.$$

Here we will neglect the angles β and γ so that we get

$$\zeta = 90^\circ - \theta \quad (4.37)$$

Based upon the above results, we can express the coefficients in Mathieu's equation as:

$$a_1 = [12.35962 \cdot E \cdot I / (\rho \cdot L^4) + 0.871558 \cdot f_{x0} / (\rho \cdot L^2)] / \omega^2 \quad (4.38a)$$

or

$$a_1 = [12.35962 \cdot E \cdot I / (\rho \cdot L^4) + 0.871558 \cdot N \cdot (\cos \zeta + \mu_0 \cdot \sin \zeta) / (\rho \cdot L^2)] / \omega^2 \quad (4.38b)$$

$$b_1 = [0.871558 \cdot f_{x1} / (\rho \cdot L^2)] / \omega^2 \quad (4.39a)$$

or

$$b_1 = [0.871558 \cdot \mu_1 \cdot N \cdot \sin \zeta / (\rho \cdot L^2)] / \omega^2 \quad (4.39b)$$

A parametric evaluation of the conditions that lead to instability is presented below.

A FORTRAN computer program called MATHIEU was written to calculate the values of the coefficients a_1 and b_1 for various values of the parameters such as the beam free length, elastic modulus, thickness, angle θ , and the coefficients of friction. A stability surface will then be generated using a range of values for a_1 and b_1 close to the calculated values. The stability surface will result in stable behavior if all the

values of S are within -2 and 2, otherwise it will be unstable. A contour plot will then be generated to produce the boundaries for stability. The boundaries are lines representing the intersection of the stability surface with a plane passing by $S=-2$ and another plane passing by $S=2$. In the contour plots, a shade region corresponds to unstable response and a clear region corresponds to stable response. The contour plot is a localized Strutt's diagram. Contour plots are presented for regions that have both stable and unstable responses.

Tables 1 and 2 present a_1 and b_1 values for both the polyurethane and steel beams based on their geometric configuration and coefficient of friction. In these tables, an instability region is marked by a shaded area. Regions that have both stable and unstable areas are splitted with one half clear for stable response and the other half shaded for unstable response. The stability plots and contour plots are shown in **figures 15-32** for polyurethane and in **figures 33-44** for steel.

From the parametric equations, it is clear that both a_1 and b_1 are greatly controlled by the value of ω . The higher ω is the smaller will be the a_1 and b_1 values. In this thesis, ω is fixed at once per revolution of the roller or 15 rad/s which is much smaller than the first natural frequency for either the axial or transverse vibrations.

The a_1 value is also controlled by the stiffness of the beam ($E \cdot I$), its mass density (ρ), its free extension (L), and to a lesser extent by the average coefficient of friction (μ_0), the normal load (N) and the inclination angle (ζ). The b_1 value is determined by the magnitude of variability in the coefficient of friction (μ_1), the normal load, the inclination angle, the mass density and free extension. Small variations of μ_0 are studied in this thesis, but higher variations are possible due to a number of factors such

as heat build up at contact, poor lubrication, and others.

The values of a_1 and b_1 are considerably higher than it has been studied in the literature. The a_1 values are considerably higher than b_1 values which by figure 13 should indicate higher stability since the regions of stability appear to widen in shape for higher a_1 values. In spite of that, the results presented in this thesis show regions of instability for higher values of the b_1 coefficients. It appears that instability regions widen for higher b_1 values and that the boundaries ($S=-2$ or $S=2$) may tilt either to the right or the left of the a_1 axis as they intersect. These changes in shape of the boundaries make it difficult to generalize the results.

It is advisable to perform an analysis close to the region under investigation and then expand the region to account for variability in the beam dimensions, in the elastic properties, in the friction, in the inclination angle and normal load. Such an analysis would probably require higher number of nodes and it would consume considerable time to complete the calculations.

The analysis indicates that stability is arrived at lower values of b_1 . Lower values of b_1 are achieved by longer beam extension and lower μ_1 . The tables also show that the above rule is not always valid and that each case may need a careful analysis to determine its stability.

TABLE 1

TABLES OF MATHIEU COEFFICIENTS FOR VARIOUS
PARAMETERS IN THE CANTILEVER BEAM

E=1000 $\gamma=0.04$ THICKNESS=0.050" WIDTH=1.0" $\mu_1=0.01\mu_0$

$\theta = 30^\circ$ N=0.1 lbf

| L | μ_0 | | μ_0 | | μ_0 | |
|-------|---------|------|---------|------|---------|-------|
| | 0.3 | | 0.9 | | 1.5 | |
| | a1 | b1 | a1 | b1 | a1 | b1 |
| 0.250 | 292.11 | 3.11 | 298.33 | 9.33 | 304.55 | 15.55 |
| 0.450 | 2976.7 | 0.86 | 3168.8 | 2.88 | 3360.8 | 4.80 |
| 0.650 | 753.9 | 0.46 | 845.9 | 1.38 | 938.0 | 2.30 |

$\theta = 30^\circ$ N=0.2 lbf

| L | μ_0 | | μ_0 | | μ_0 | |
|-------|---------|------|---------|-------|---------|-------|
| | 0.3 | | 0.9 | | 1.5 | |
| | a1 | b1 | a1 | b1 | a1 | b1 |
| 0.250 | 30120 | 6.22 | 31365 | 18.66 | 32609 | 31.11 |
| 0.450 | 3257.5 | 1.92 | 3641.6 | 5.76 | 4025.7 | 9.60 |
| 0.650 | 888.5 | 0.92 | 1072.6 | 2.76 | 1256.6 | 4.60 |

$\theta = 45^\circ$ N=0.1 lbf

| L | μ_0 | | μ_0 | | μ_0 | |
|-------|---------|-------|---------|------|---------|-------|
| | 0.3 | | 0.9 | | 1.5 | |
| | a1 | b1 | a1 | b1 | a1 | b1 |
| 0.250 | 29401 | 2.54 | 29910 | 7.62 | 30418 | 12.70 |
| 0.450 | 3035.7 | 0.784 | 3192.4 | 2.35 | 3349.2 | 3.92 |
| 0.650 | 782.1 | 0.38 | 857.2 | 1.13 | 932.4 | 1.88 |

$\theta = 45^\circ$ N=0.2 lbf

| L | μ_0 | | μ_0 | | μ_0 | |
|-------|---------|------|---------|-------|---------|-------|
| | 0.3 | | 0.9 | | 1.5 | |
| | a1 | b1 | a1 | b1 | a1 | b1 |
| 0.250 | 30502 | 5.08 | 31518 | 15.24 | 32534 | 25.40 |
| 0.450 | 3375.4 | 1.57 | 3689 | 4.70 | 4002.6 | 7.84 |
| 0.650 | 945 | 0.75 | 1095.3 | 2.25 | 1245.6 | 3.76 |

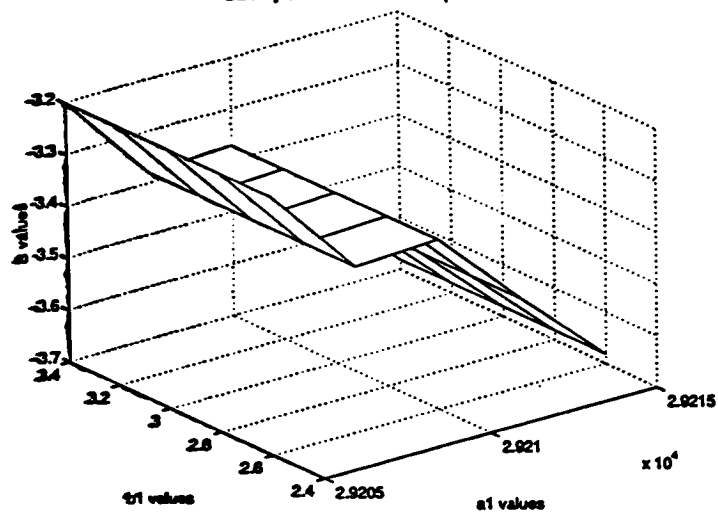
$\theta = 60^\circ$ N=0.1 lbf

| L | μ_0 | | μ_0 | | μ_0 | |
|-------|---------|------|---------|------|---------|------|
| | 0.3 | | 0.9 | | 1.5 | |
| | a1 | b1 | a1 | b1 | a1 | b1 |
| 0.250 | 29517 | 1.80 | 29877 | 5.39 | 30236 | 8.98 |
| 0.450 | 3071.4 | 0.55 | 3182.3 | 1.66 | 3293 | 2.77 |
| 0.650 | 799.3 | 0.27 | 852.4 | 0.80 | 905.5 | 1.32 |

$\theta = 60^\circ$ N=0.2 lbf

| L | μ_0 | | μ_0 | | μ_0 | |
|-------|---------|------|---------|-------|---------|-------|
| | 0.3 | | 0.9 | | 1.5 | |
| | a1 | b1 | a1 | b1 | a1 | b1 |
| 0.250 | 30734 | 3.59 | 31452 | 10.77 | 32171 | 17.96 |
| 0.450 | 3446.9 | 1.11 | 3668.7 | 3.32 | 3890.4 | 5.54 |
| 0.650 | 979.2 | 0.53 | 1085.5 | 1.59 | 1194.8 | 2.66 |

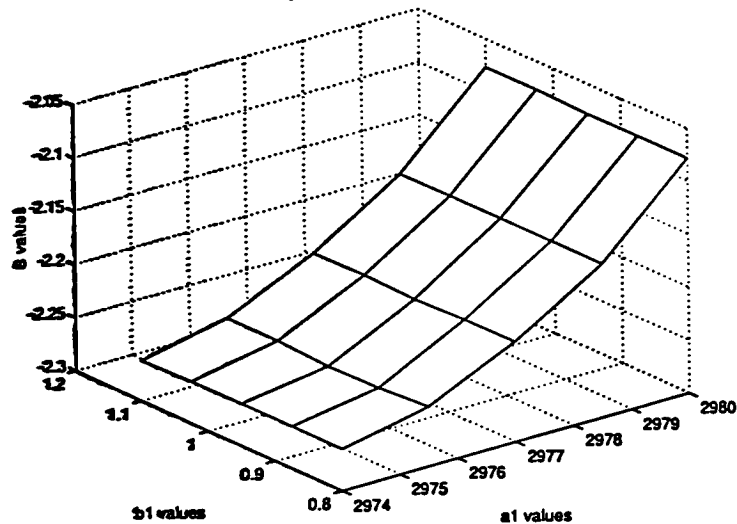
Stability Surface for Mathieu Equation



$\Theta = 30^\circ$ $N = 0.116$ $L = 0.250^\circ$ $\mu_0 = 0.3$
UNSTABLE

UNSTABLE

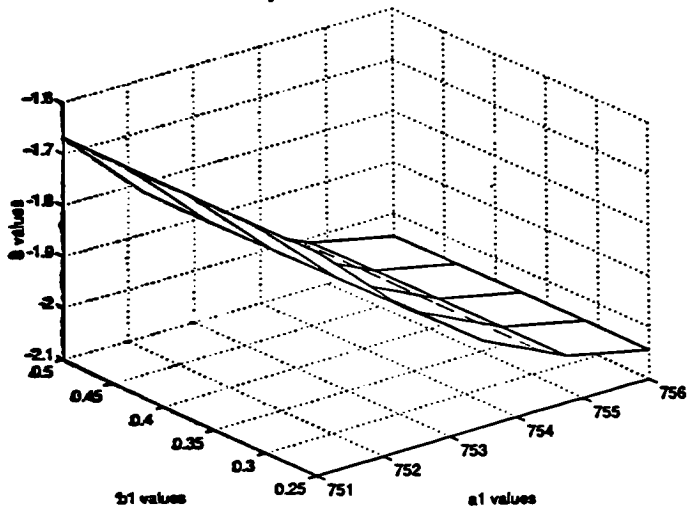
Stability Surface for Mathieu Equation



$\Theta = 30^\circ$ $N = 0.116$ $L = 0.450^\circ$ $\mu_0 = 0.3$
UNSTABLE

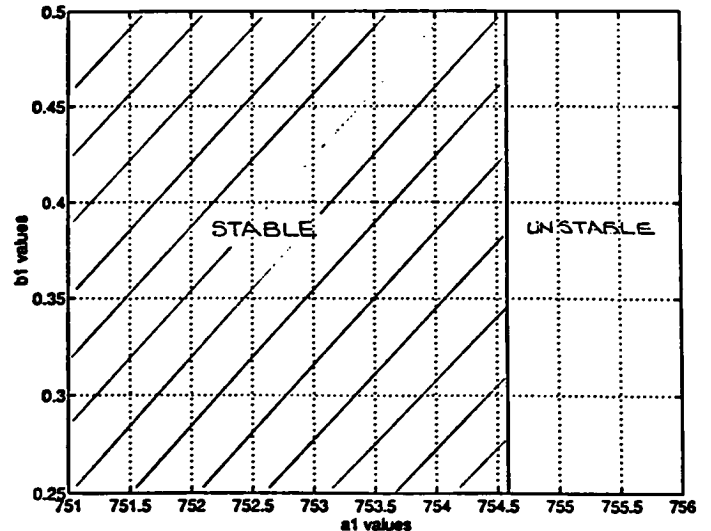
UNSTABLE

Stability Surface for Mathieu Equation



$\Theta = 30^\circ$ $N = 0.116$ $L = 0.650^\circ$ $\mu_0 = 0.3$
STABLE FOR $a_1 < 754.5$

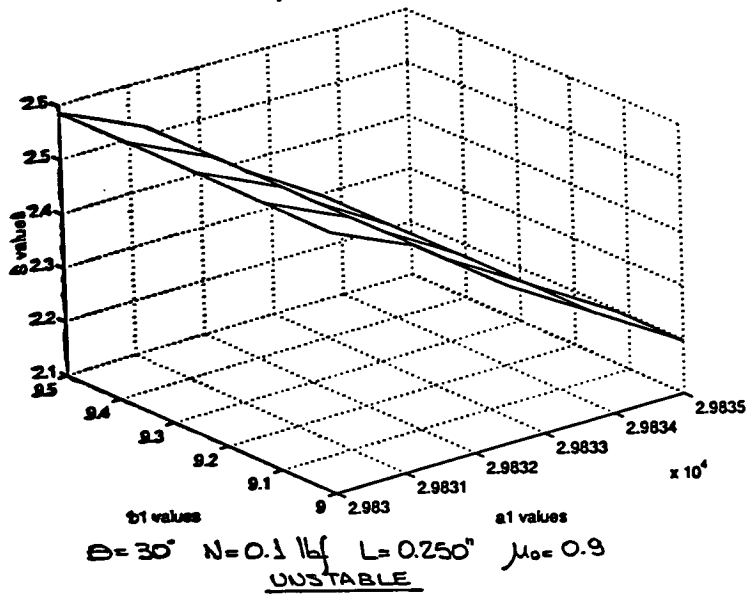
Contour Plot for $S=-2$ or $S=2$



$\Theta = 30^\circ$ $N = 0.116$ $L = 0.650^\circ$ $\mu_0 = 0.3$

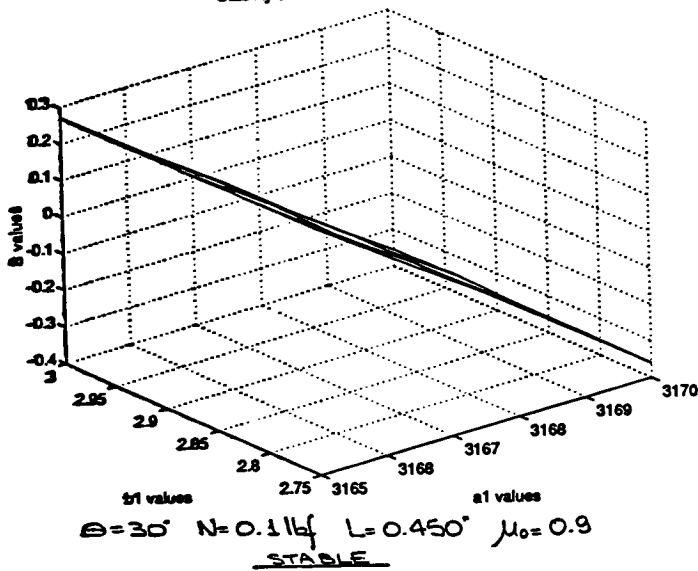
FIGURE 15

Stability Surface for Mathieu Equation



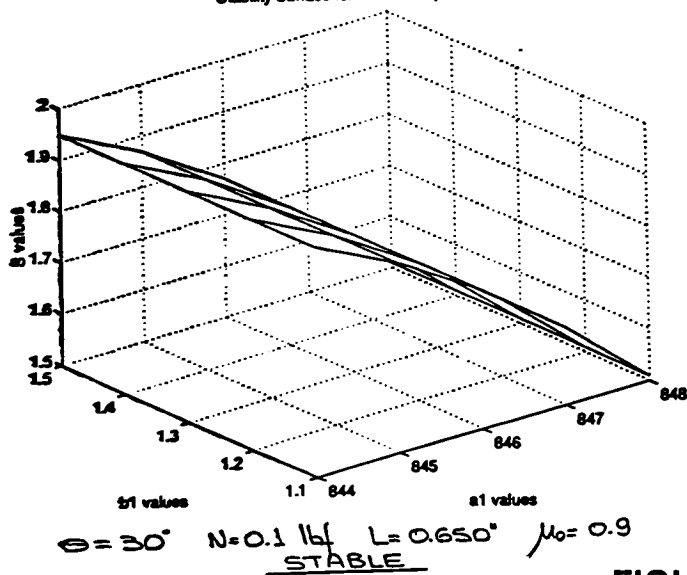
UNSTABLE

Stability Surface for Mathieu Equation



STABLE

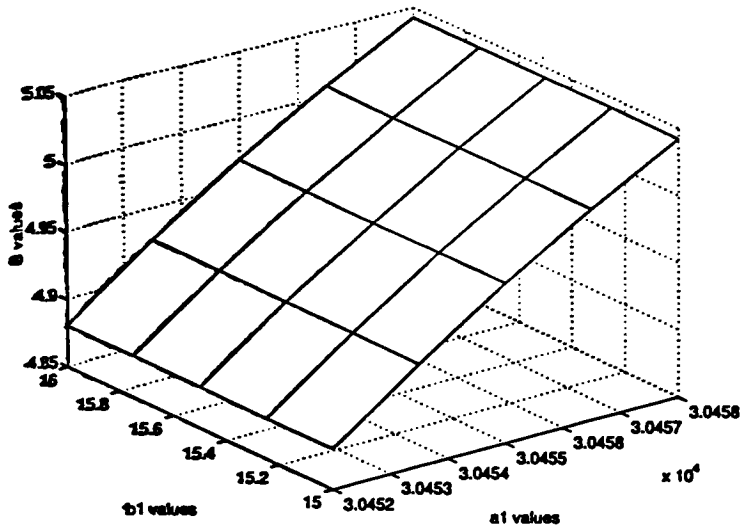
Stability Surface for Mathieu Equation



STABLE

FIGURE 16

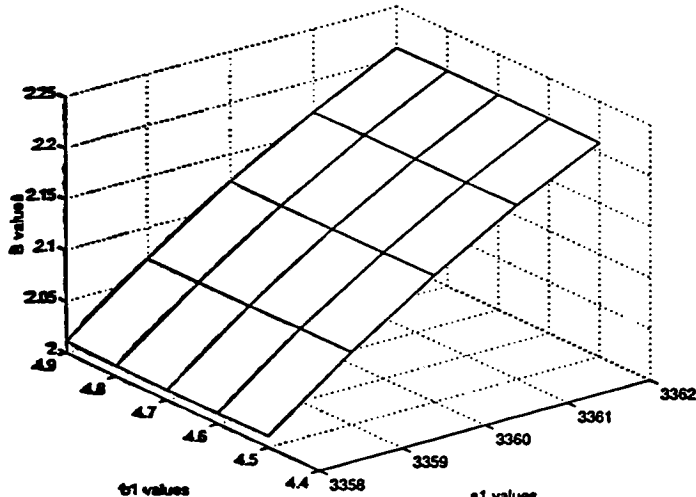
Stability Surface for Mathieu Equation



$\Theta = 30^\circ$ $N = 0.116$ $L = 0.250^\circ$ $\mu_0 = 1.5$
UNSTABLE

UNSTABLE

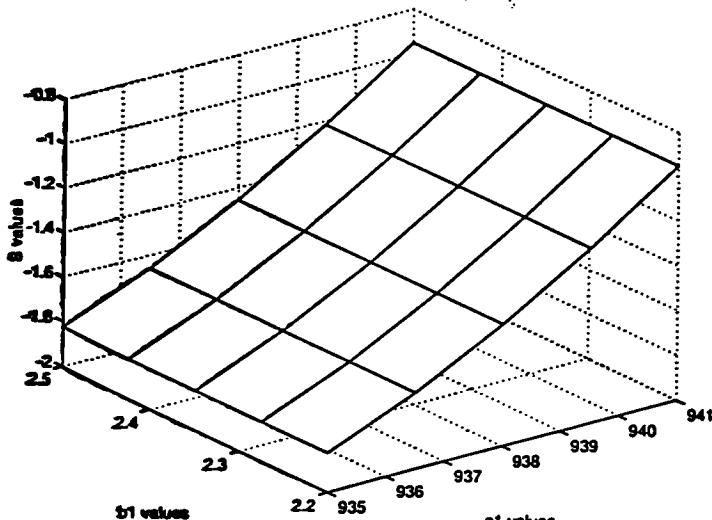
Stability Surface for Mathieu Equation



$\Theta = 30^\circ$ $N = 0.116$ $L = 0.450^\circ$ $\mu_0 = 1.5$
UNSTABLE

UNSTABLE

Stability Surface for Mathieu Equation

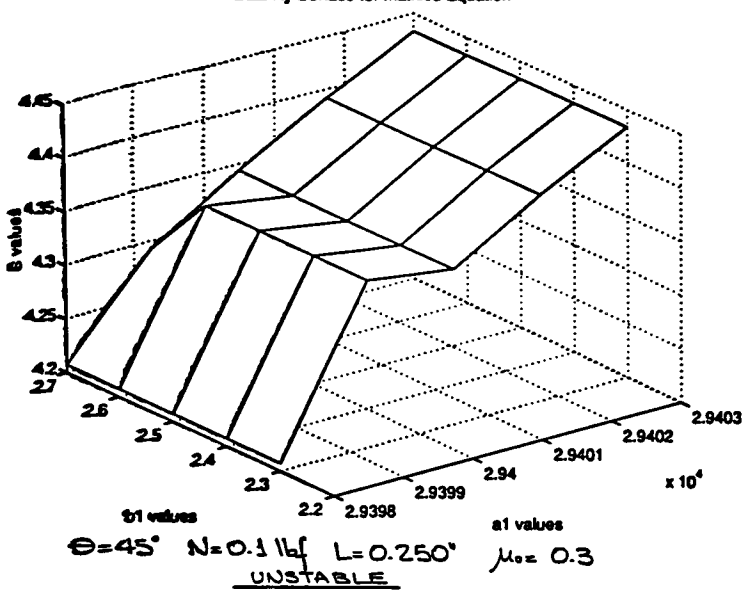


$\Theta = 30^\circ$ $N = 0.116$ $L = 0.650^\circ$ $\mu_0 = 1.5$
STABLE

STABLE

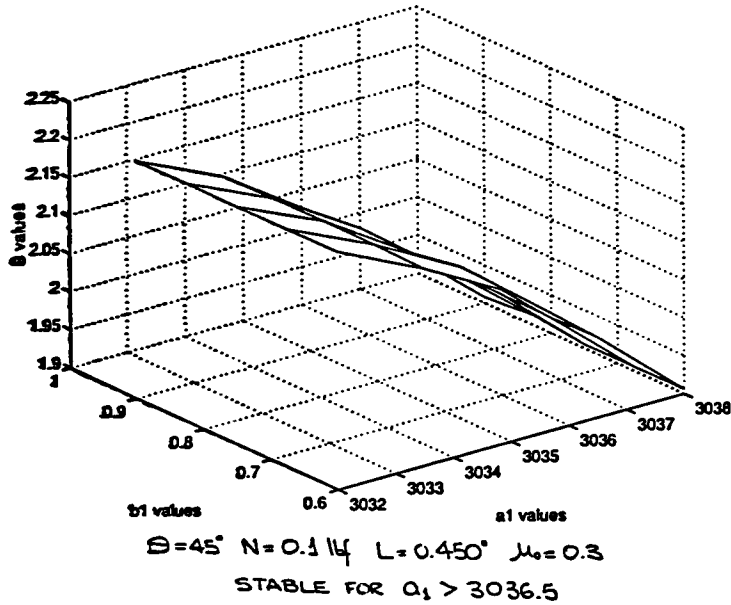
FIGURE 17

Stability Surface for Mathieu Equation

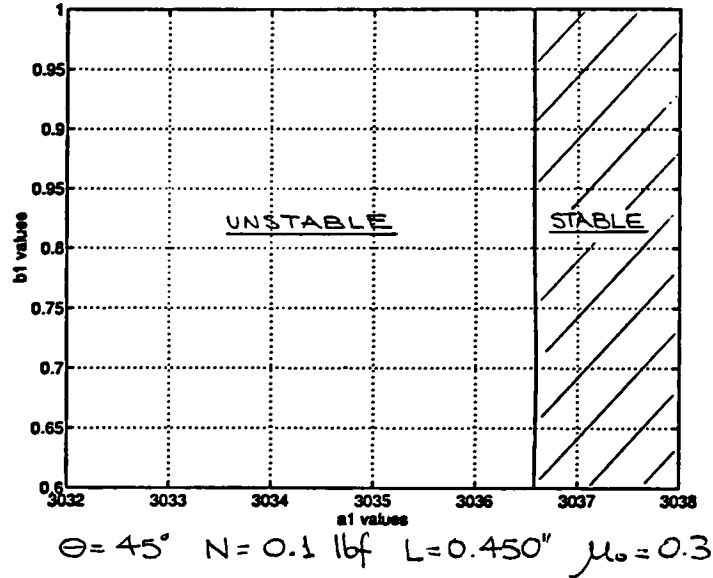


UNSTABLE

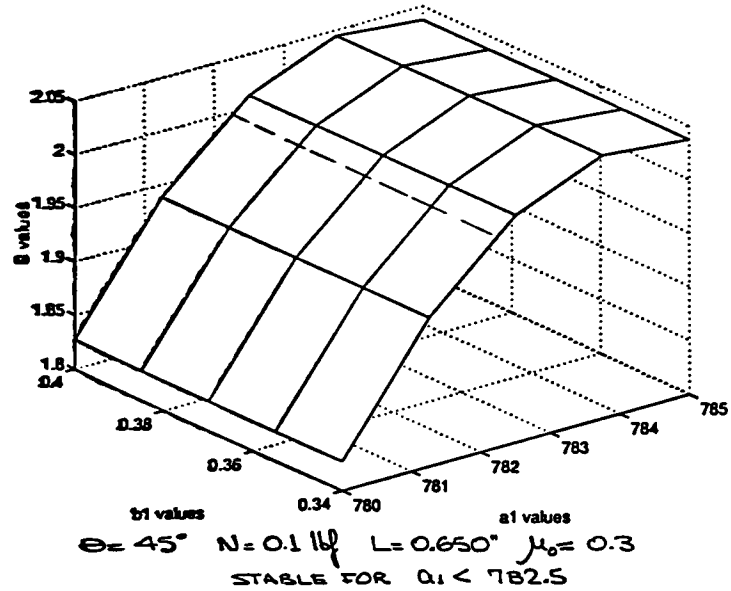
Stability Surface for Mathieu Equation



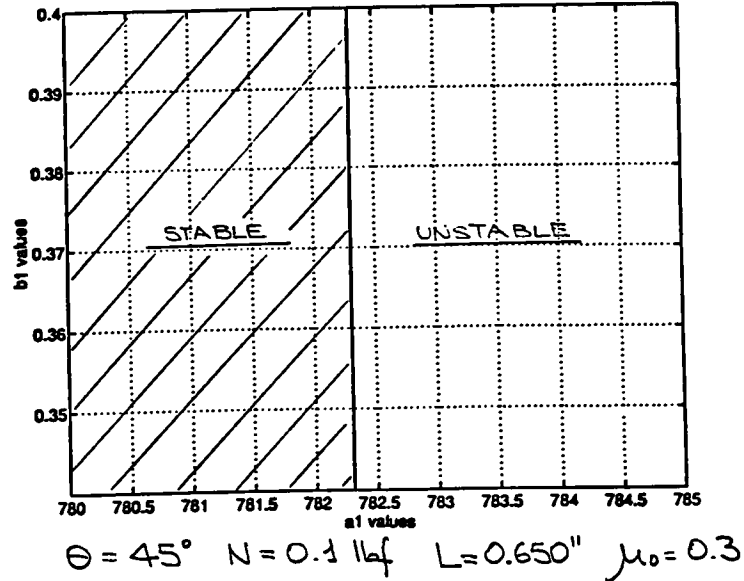
Contour Plot for $S=-2$ or $S=2$



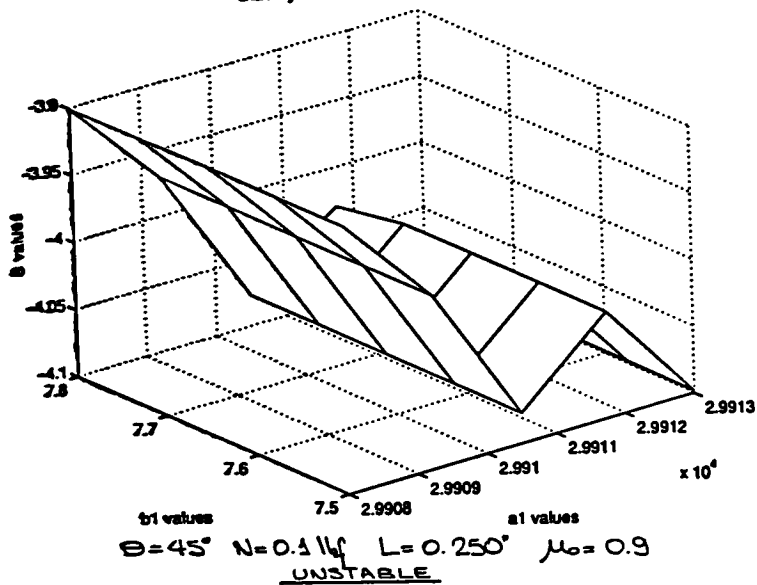
Stability Surface for Mathieu Equation



Contour Plot for $S=-2$ or $S=2$

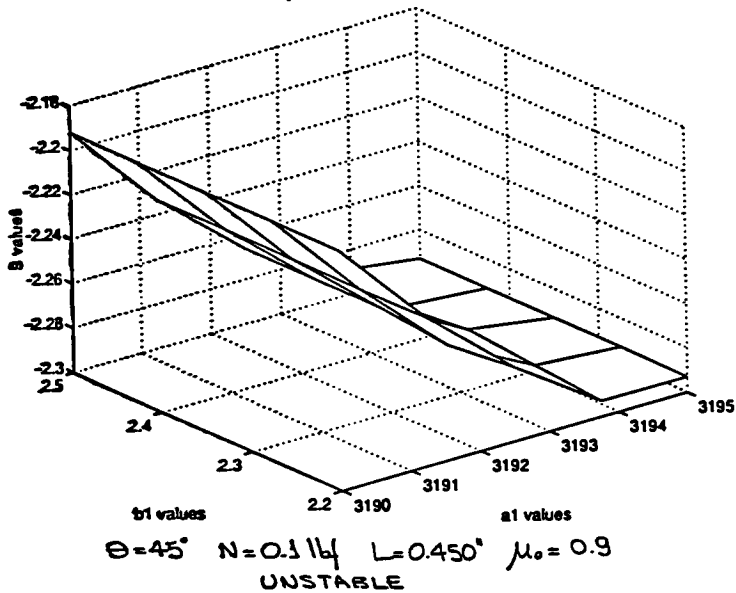


Stability Surface for Mathieu Equation



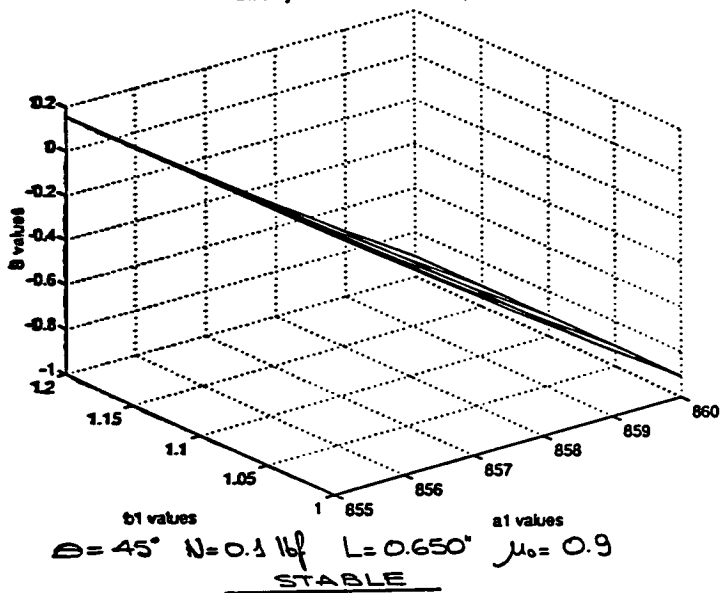
UNSTABLE

Stability Surface for Mathieu Equation



UNSTABLE

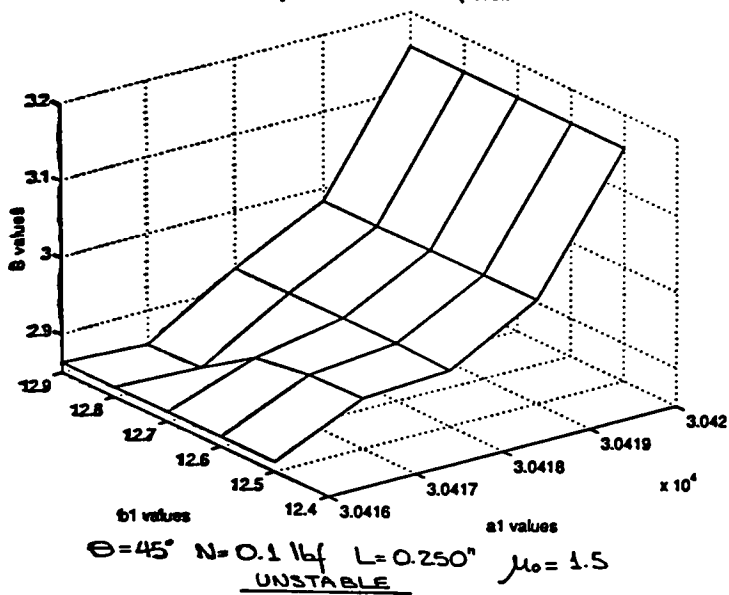
Stability Surface for Mathieu Equation



STABLE

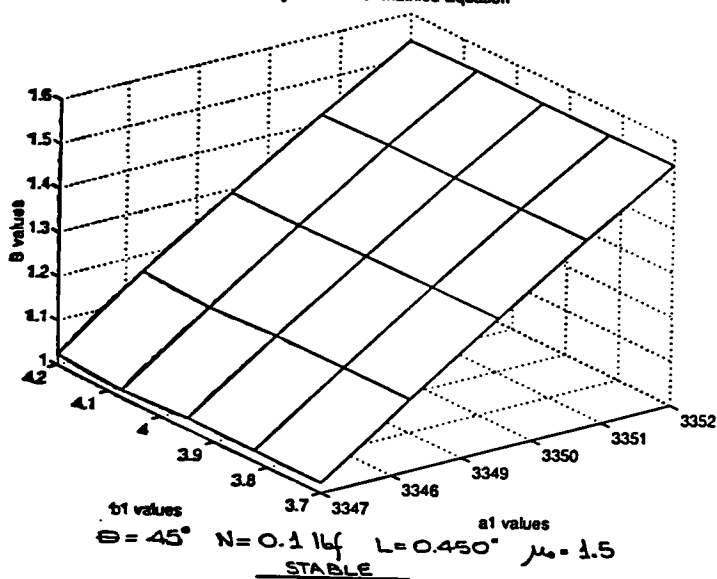
FIGURE 19

Stability Surface for Mathieu Equation



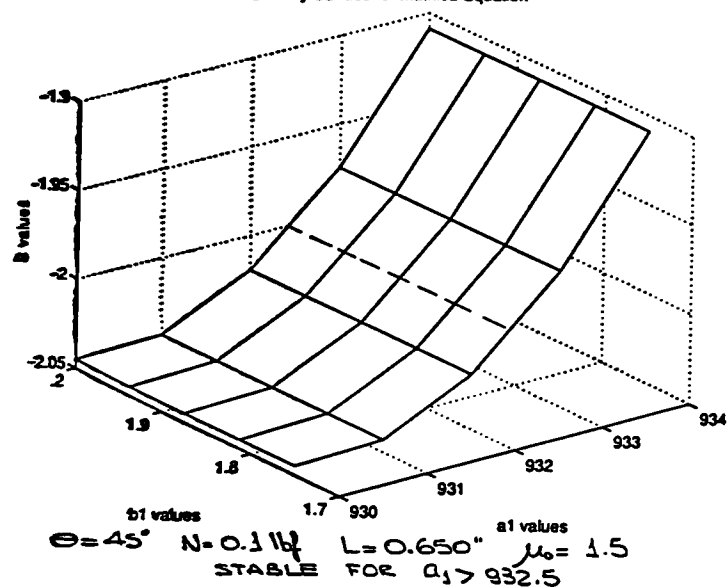
UNSTABLE

Stability Surface for Mathieu Equation



STABLE

Stability Surface for Mathieu Equation



Contour Plot for $S=-2$ or $S=2$

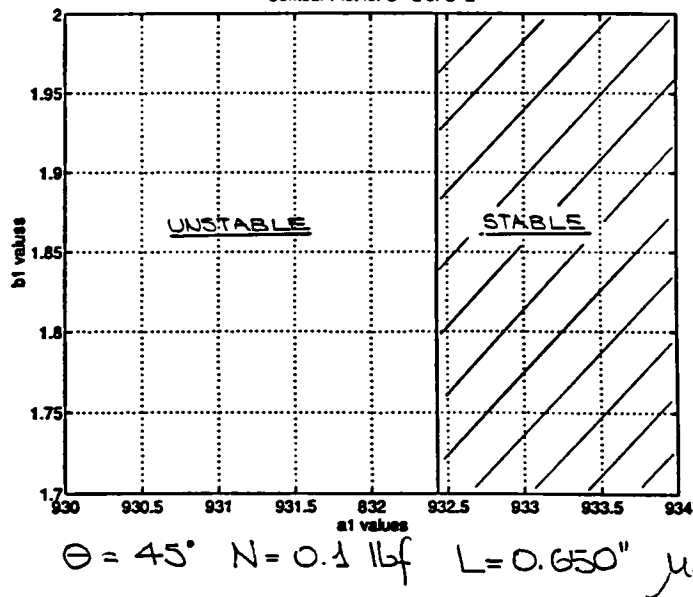
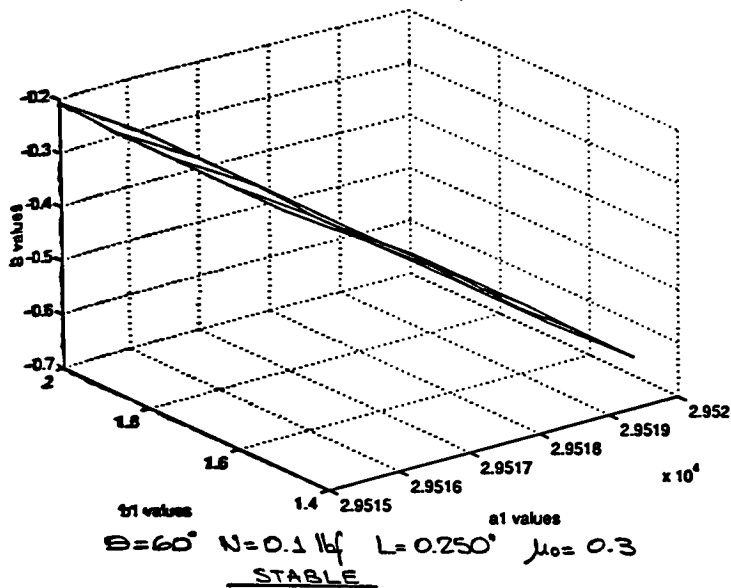


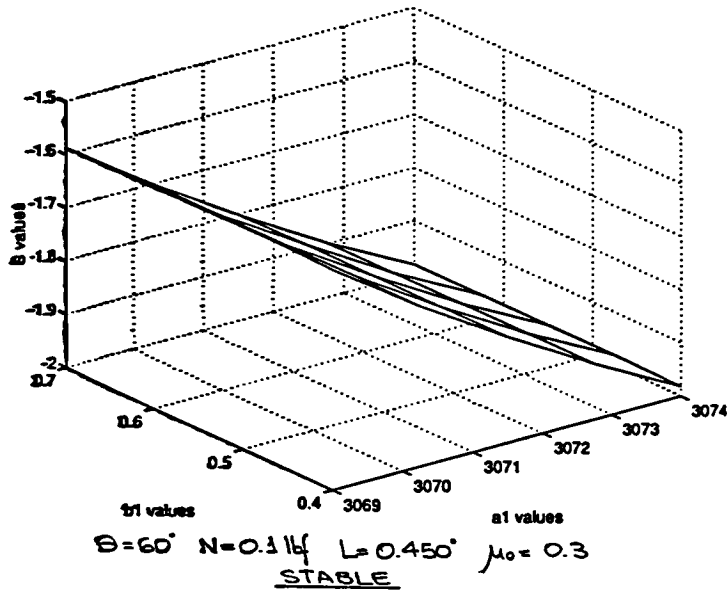
FIGURE 20

Stability Surface for Mathieu Equation



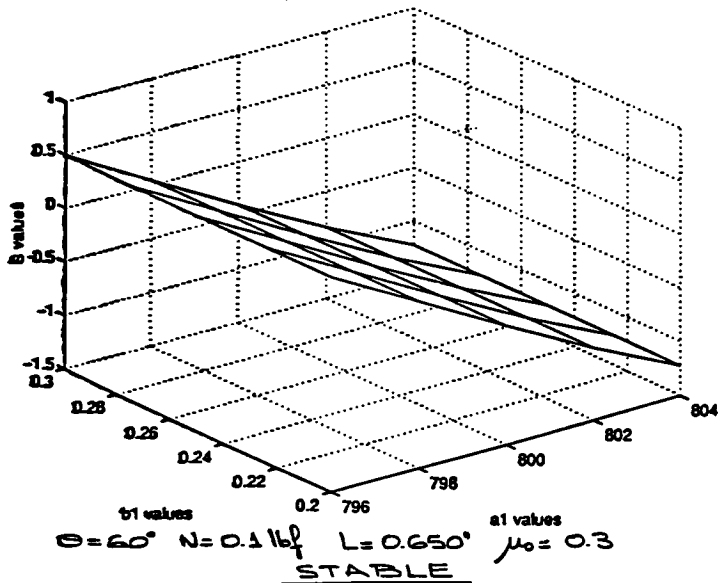
STABLE

Stability Surface for Mathieu Equation



STABLE

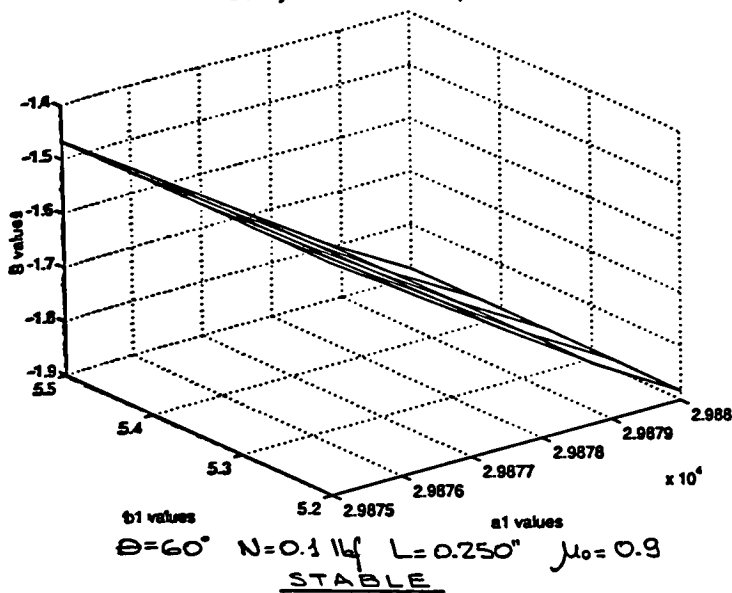
Stability Surface for Mathieu Equation



STABLE

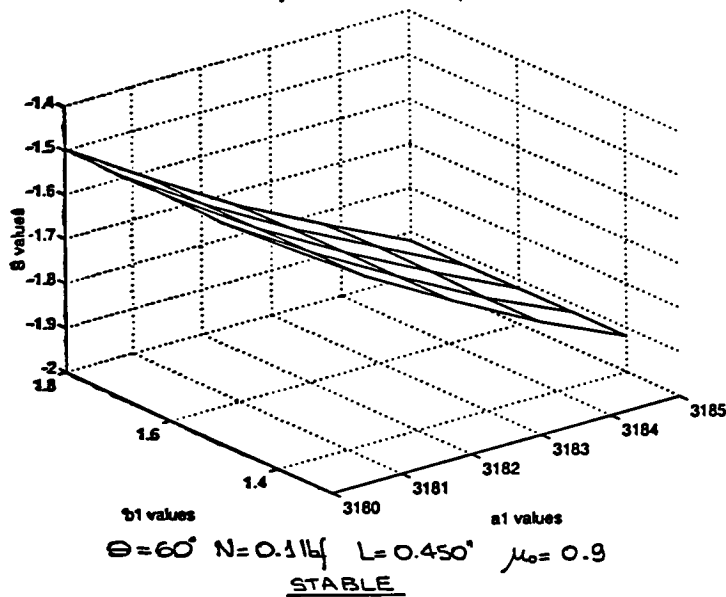
FIGURE 21

Stability Surface for Mathieu Equation



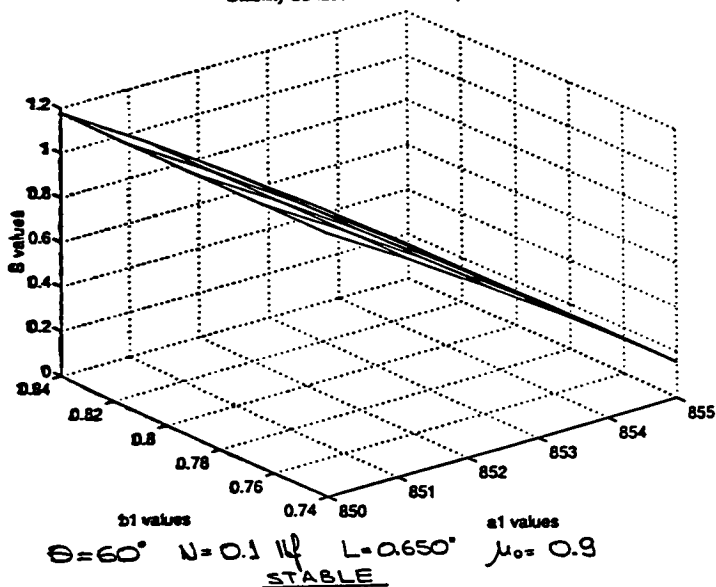
STABLE

Stability Surface for Mathieu Equation



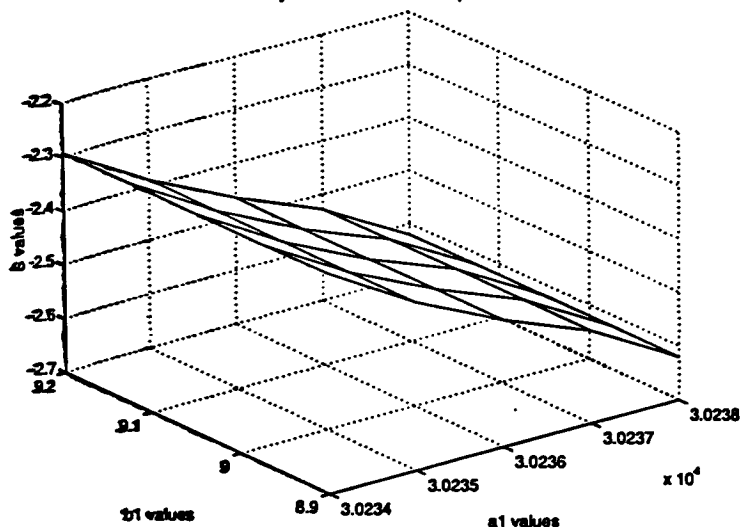
STABLE

Stability Surface for Mathieu Equation



STABLE

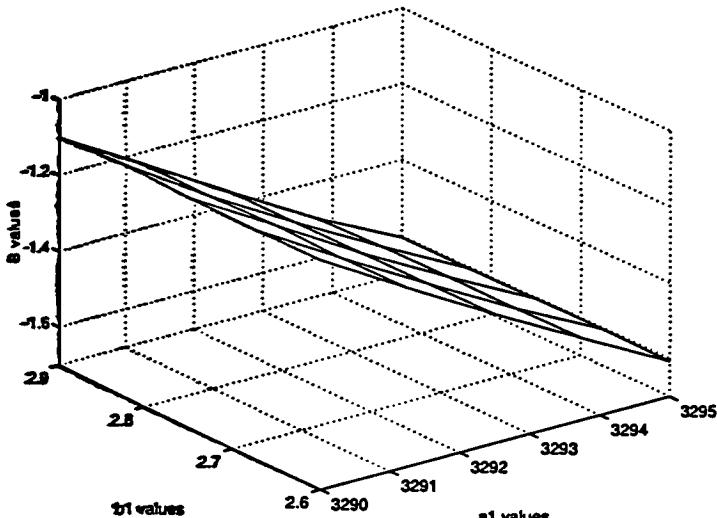
Stability Surface for Mathieu Equation



$\Theta = 60^\circ$ $N = 0.114$ $L = 0.250^\circ$ $\mu_0 = 1.5$
UNSTABLE

UNSTABLE

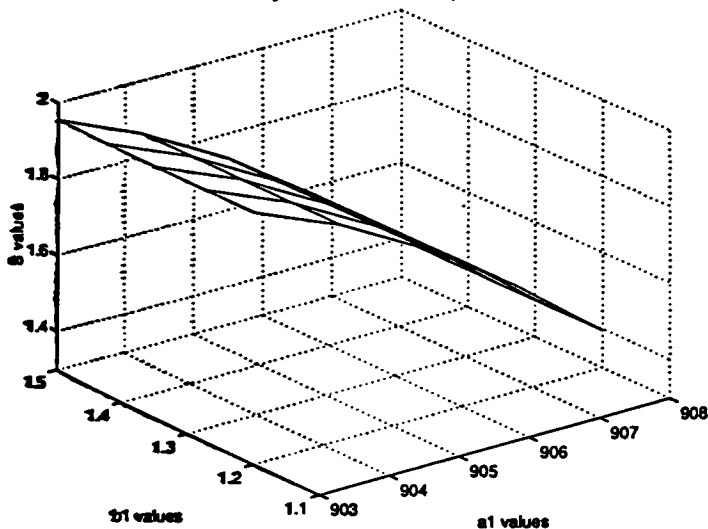
Stability Surface for Mathieu Equation



$\Theta = 60^\circ$ $N = 0.114$ $L = 0.450^\circ$ $\mu_0 = 1.5$
STABLE

STABLE

Stability Surface for Mathieu Equation

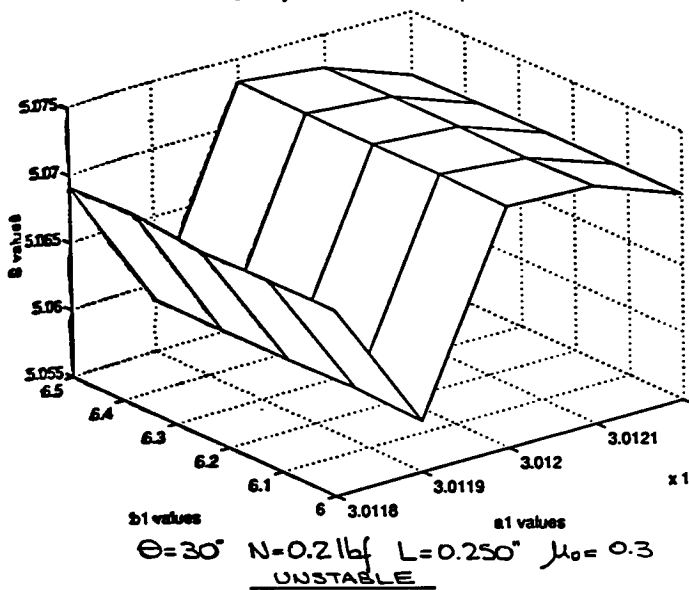


$\Theta = 60^\circ$ $N = 0.114$ $L = 0.650^\circ$ $\mu_0 = 1.5$
STABLE

STABLE

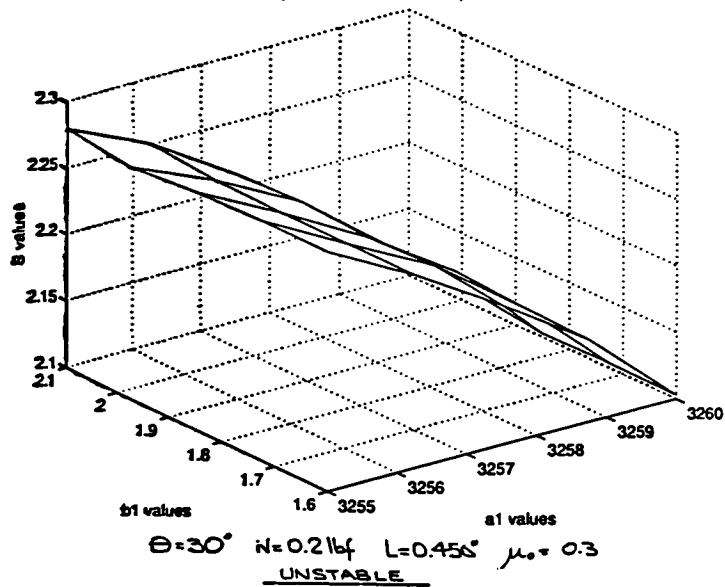
FIGURE 23

Stability Surface for Mathieu Equation



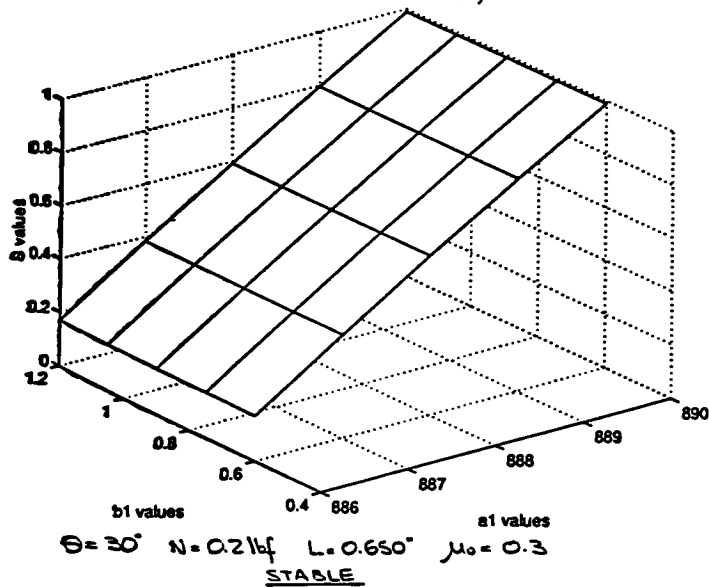
UNSTABLE

Stability Surface for Mathieu Equation



UNSTABLE

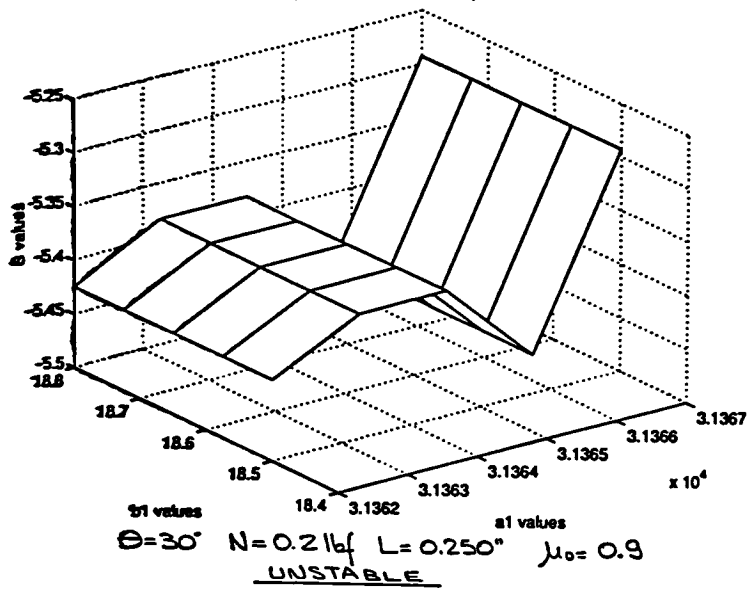
Stability Surface for Mathieu Equation



STABLE

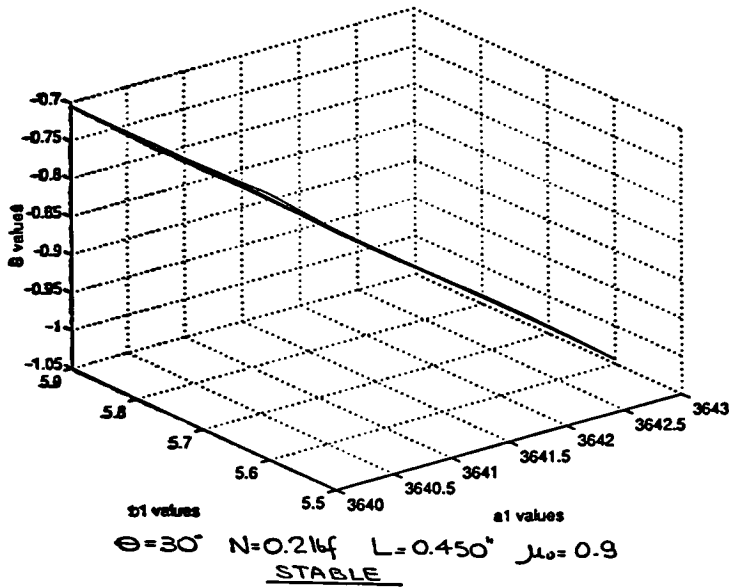
FIGURE 24

Stability Surface for Mathieu Equation



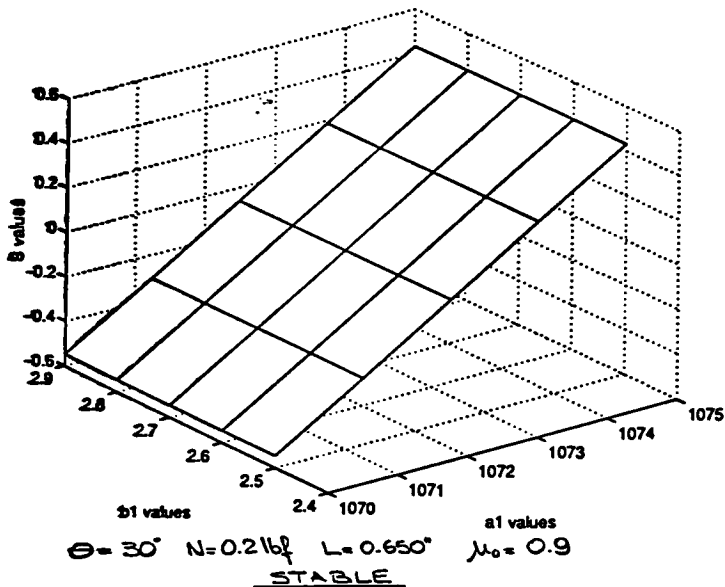
UNSTABLE

Stability Surface for Mathieu Equation



STABLE

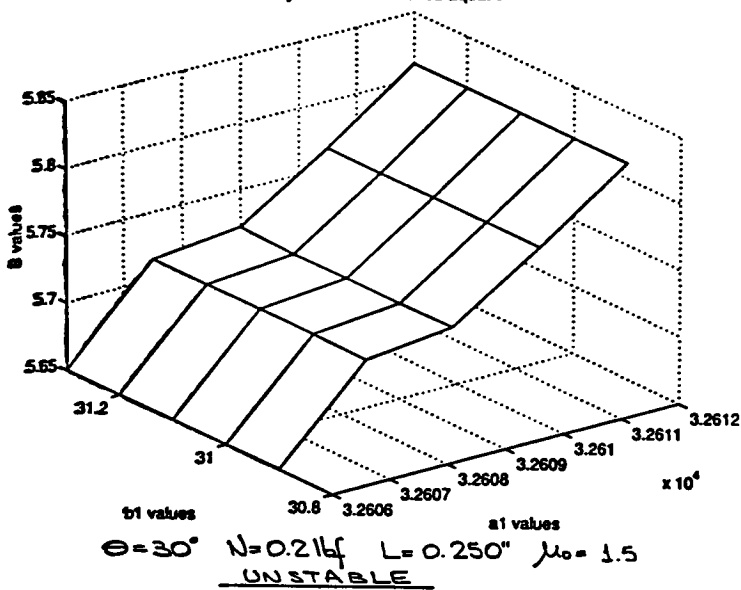
Stability Surface for Mathieu Equation



STABLE

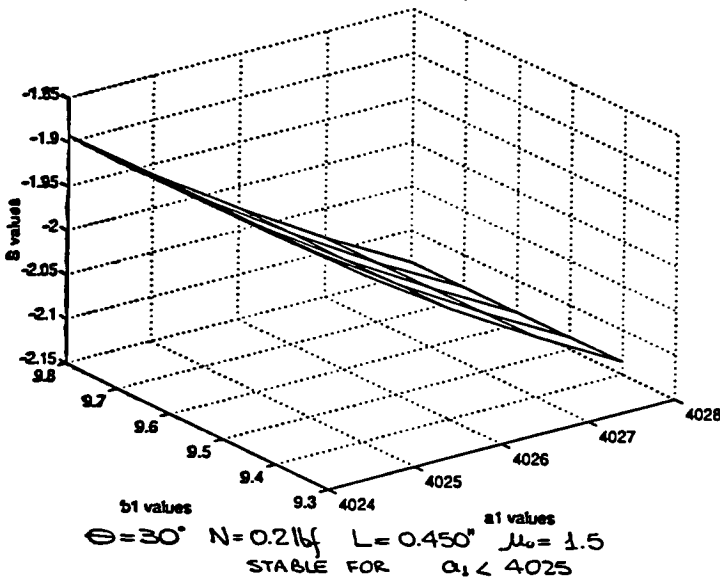
FIGURE 25

Stability Surface for Mathieu Equation

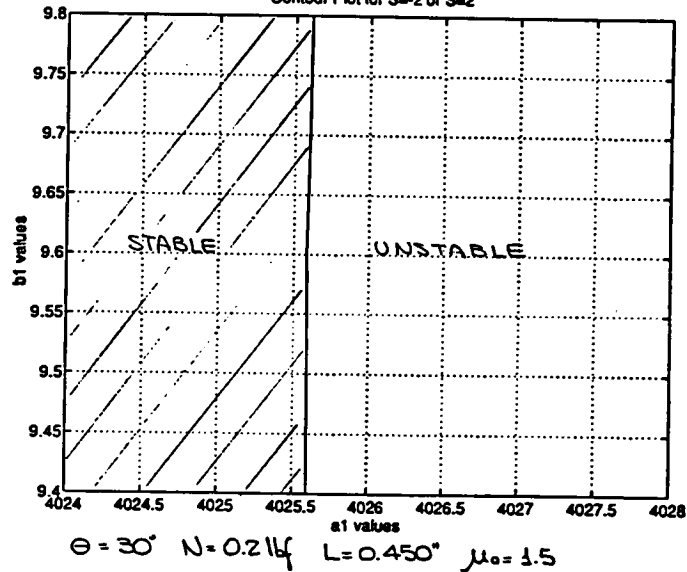


UNSTABLE

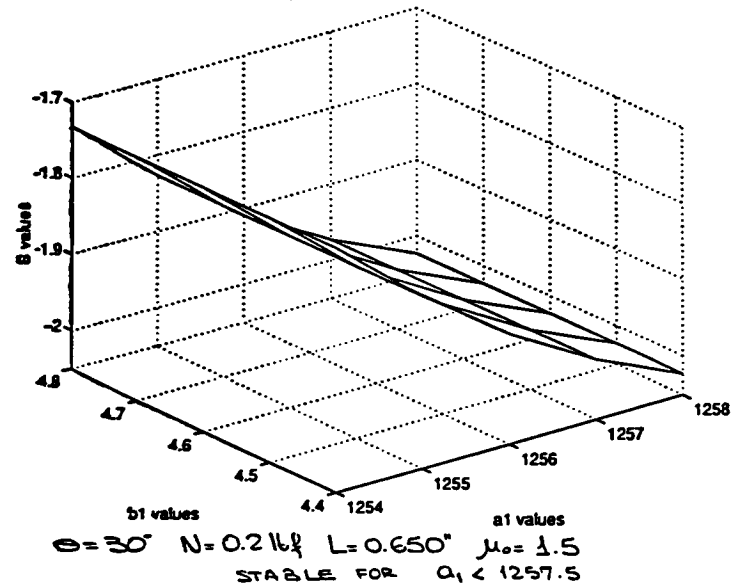
Stability Surface for Mathieu Equation



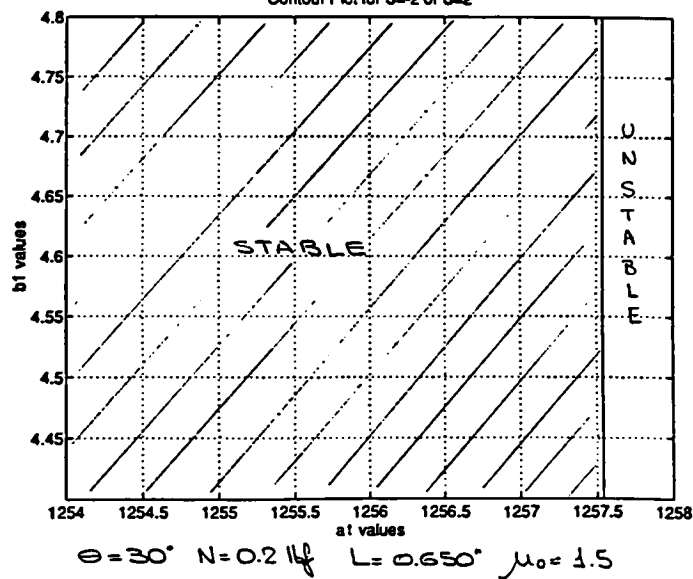
Contour Plot for $S=-2$ or $S=2$



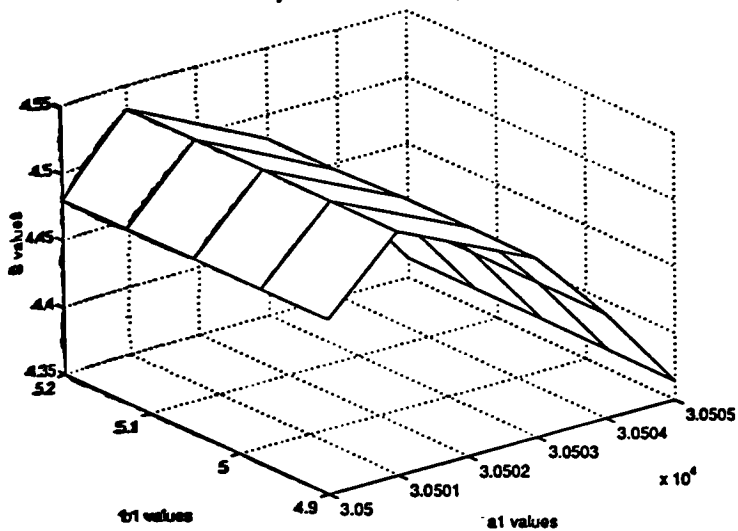
Stability Surface for Mathieu Equation



Contour Plot for $S=-2$ or $S=2$



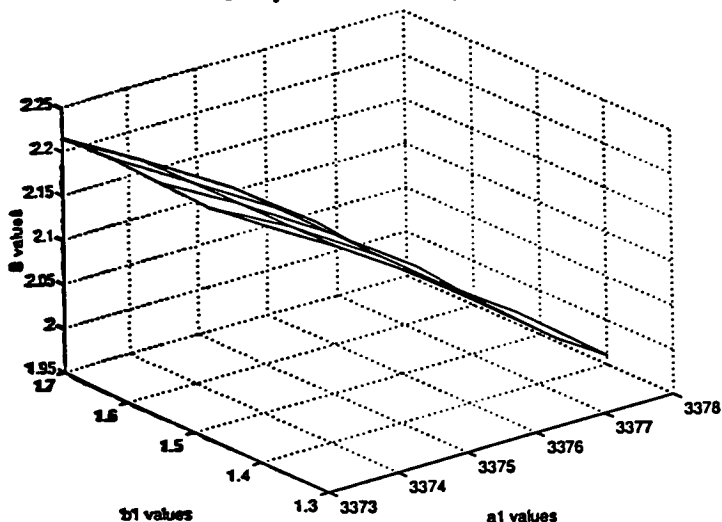
Stability Surface for Mathieu Equation



$\Theta = 45^\circ$ $N = 0.214$ $L = .250^\circ$ $\mu_0 = 0.3$
UNSTABLE

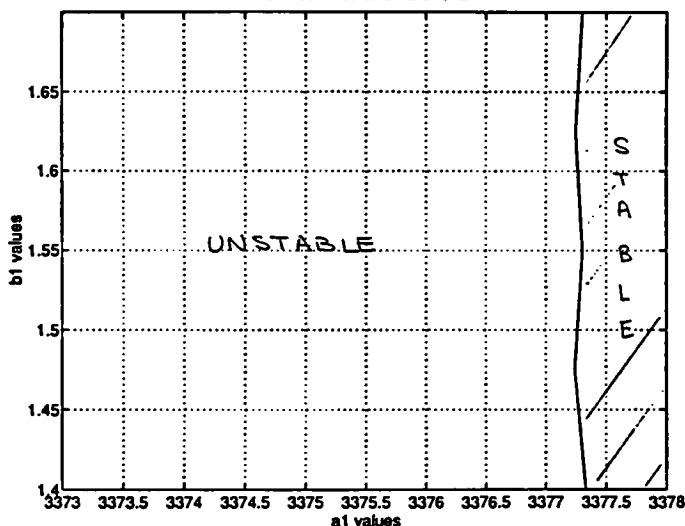
UNSTABLE

Stability Surface for Mathieu Equation



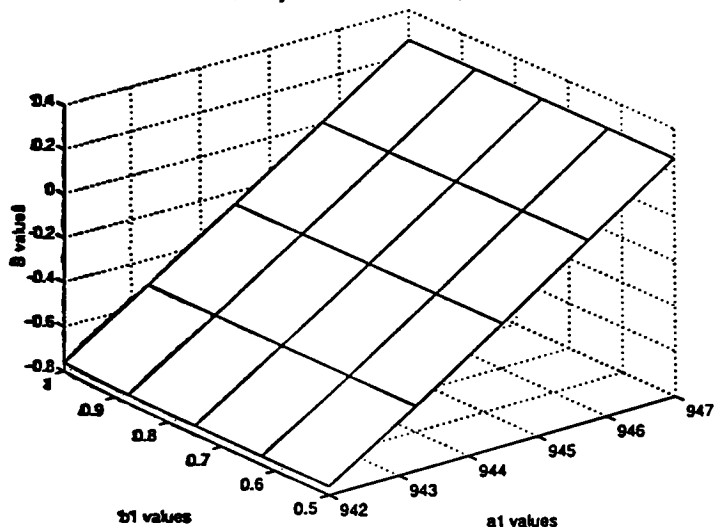
$\Theta = 45^\circ$ $N = 0.214$ $L = 0.450^\circ$ $\mu_0 = 0.3$
 STABLE FOR $Q_1 > 3377$

Contour Plot for $S=-2$ or $S=2$



$\Theta = 45^\circ$ $N = 0.214$ $L = 0.450^\circ$ $\mu_0 = 0.3$

Stability Surface for Mathieu Equation

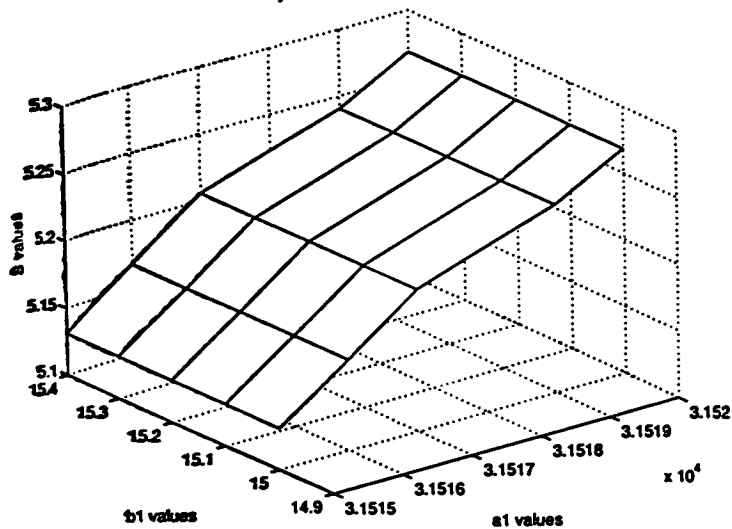


$\Theta = 45^\circ$ $N = 0.214$ $L = 0.650^\circ$ $\mu_0 = 0.3$
STABLE

STABLE

FIGURE 27

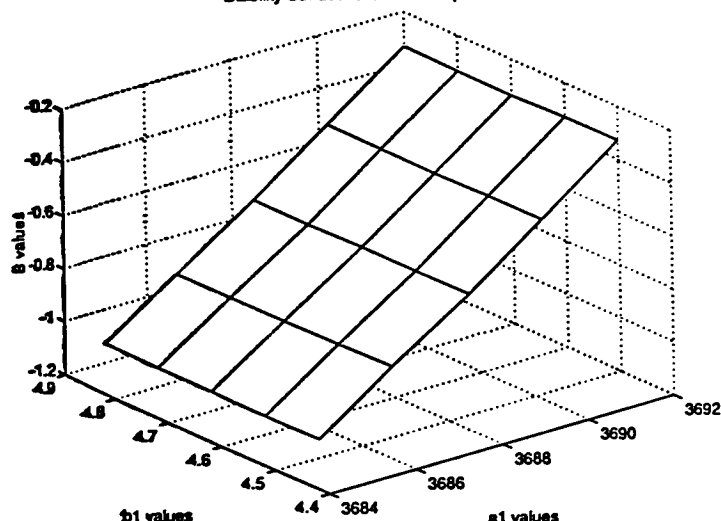
Stability Surface for Mathieu Equation



$\Theta = 45^\circ$ $N = 0.216$ $L = 0.250^\circ$ $\mu_0 = 0.9$
UNSTABLE

UNSTABLE

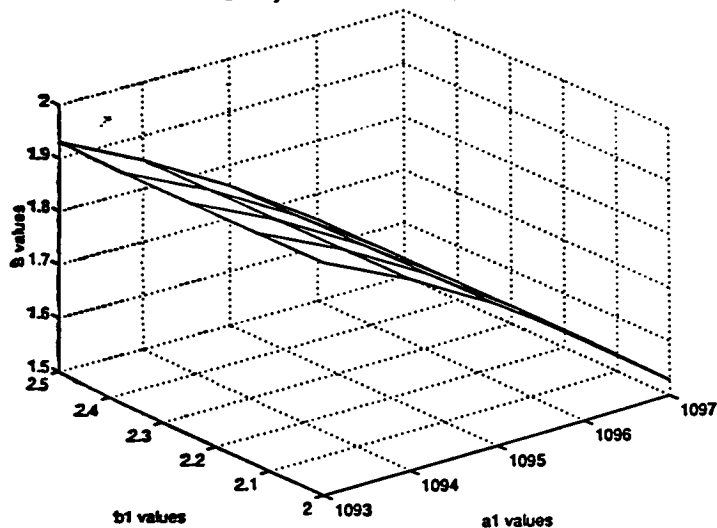
Stability Surface for Mathieu Equation



$\Theta = 45^\circ$ $N = 0.216$ $L = 0.450^\circ$ $\mu_0 = 0.9$
STABLE

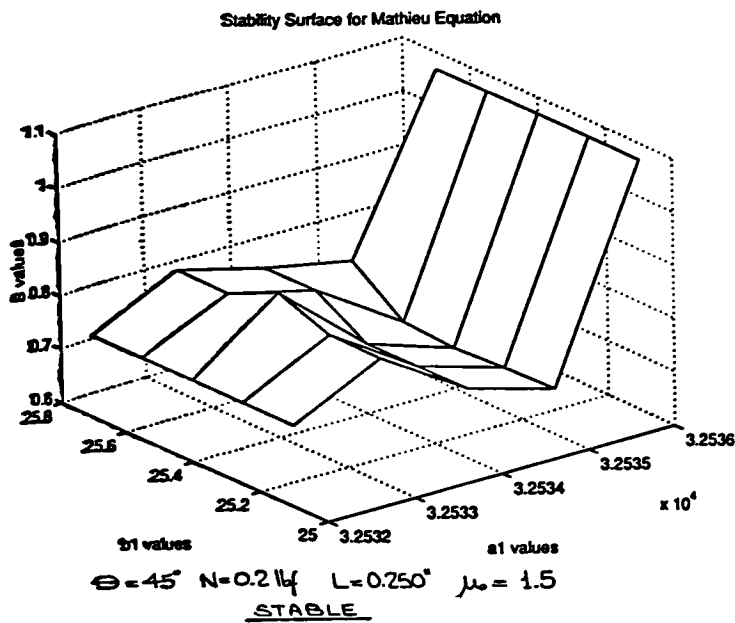
STABLE

Stability Surface for Mathieu Equation

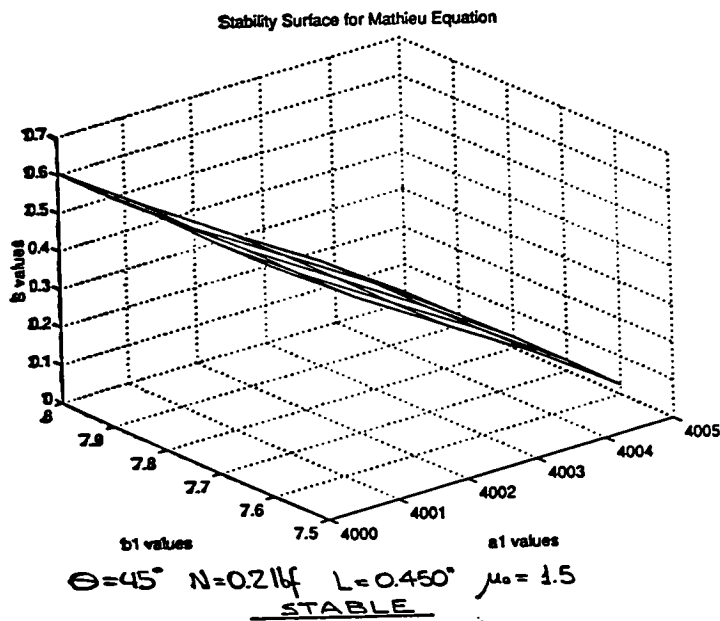


$\Theta = 45^\circ$ $N = 0.216$ $L = 0.650^\circ$ $\mu_0 = 0.9$
STABLE

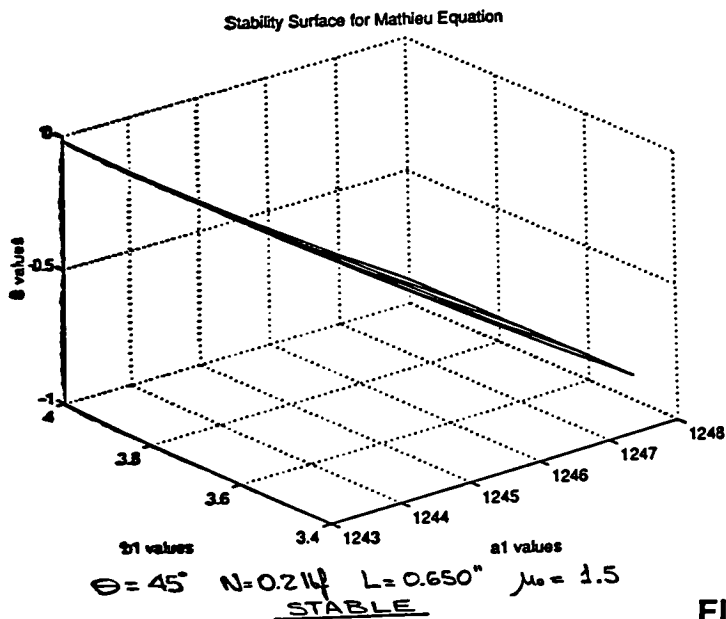
STABLE



STABLE

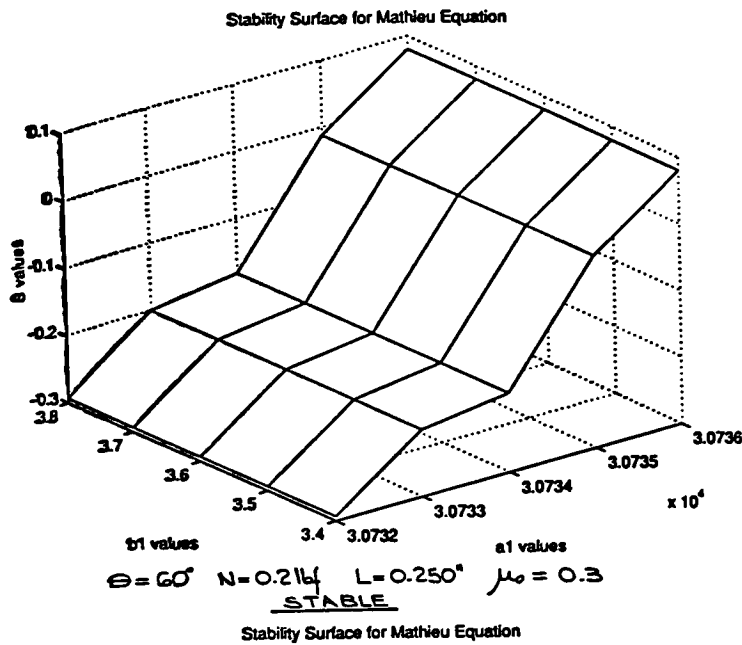


STABLE

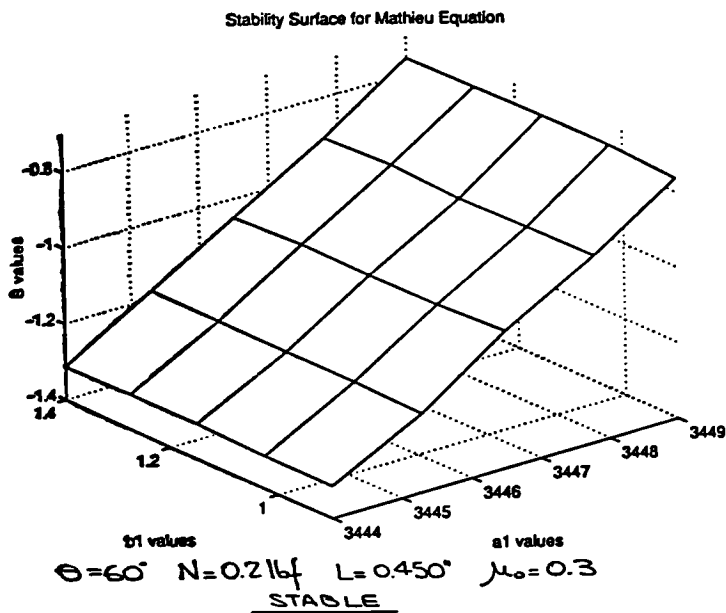


STABLE

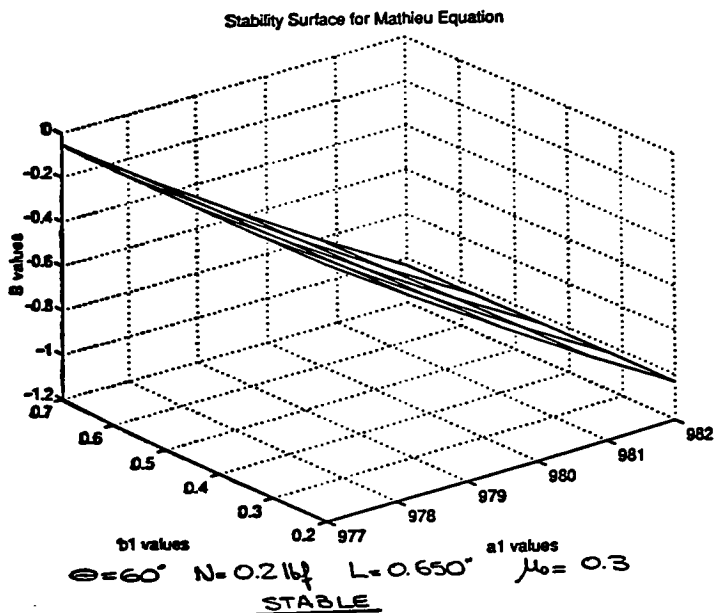
FIGURE 29



STABLE



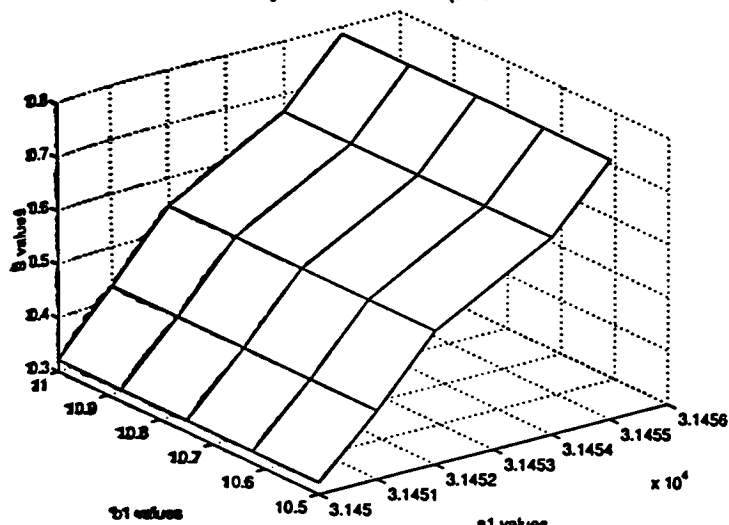
STABLE



STABLE

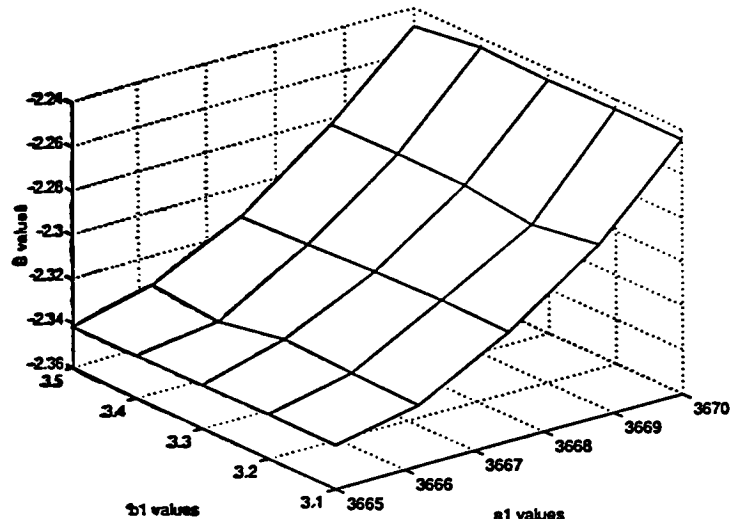
FIGURE 30

Stability Surface for Mathieu Equation



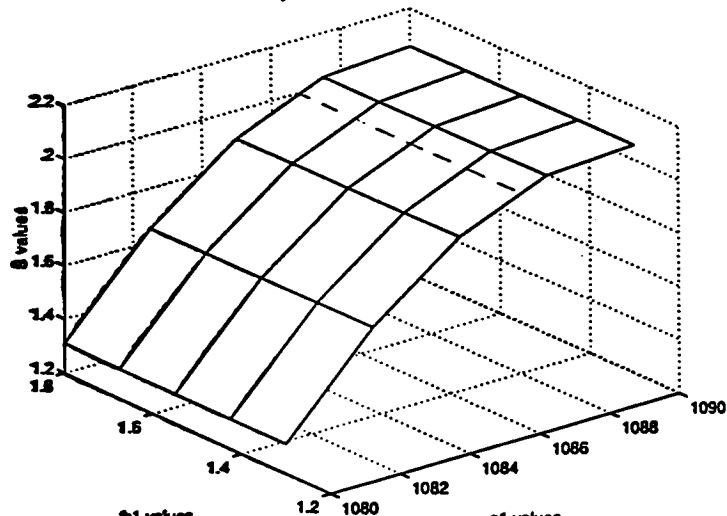
$\theta = 60^\circ$ $N = 0.216$ $L = 0.250$ $\mu_0 = 0.9$
STABLE

Stability Surface for Mathieu Equation



$\theta = 60^\circ$ $N = 0.216$ $L = 0.450$ $\mu_0 = 0.9$
UNSTABLE

Stability Surface for Mathieu Equation

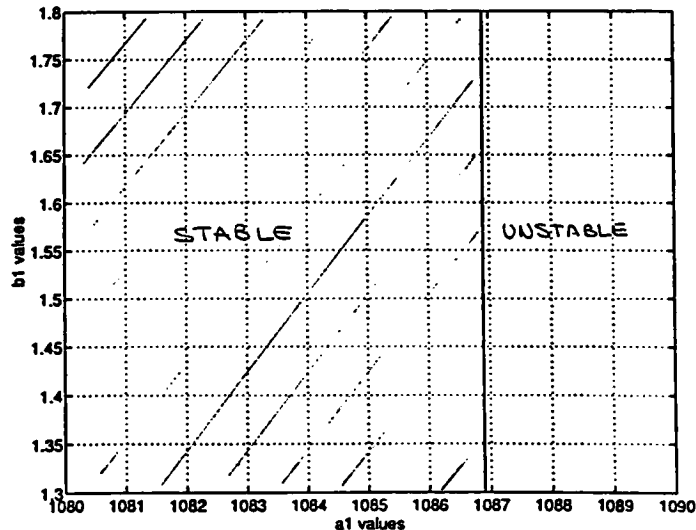


$\theta = 60^\circ$ $N = 0.216$ $L = 0.650$ $\mu_0 = 0.9$
 STABLE FOR $a_1 < 1087$

STABLE

UNSTABLE

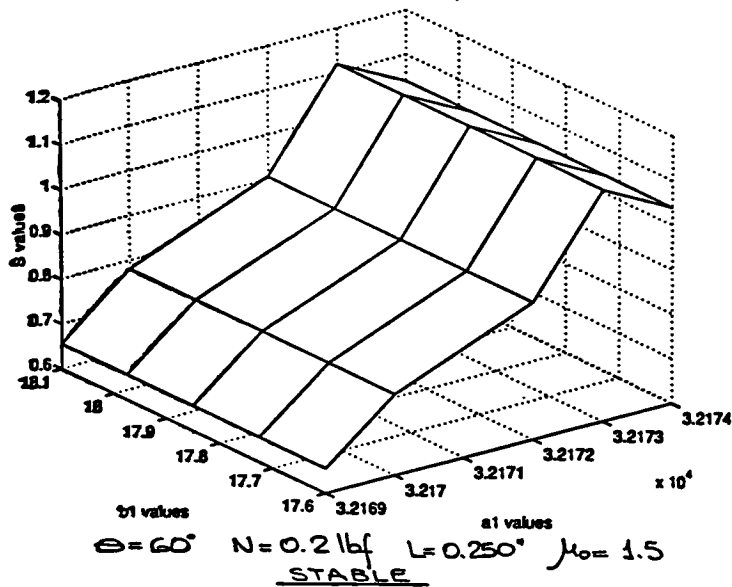
Contour Plot for $S=-2$ or $S=2$



$\theta = 60^\circ$ $N = 0.216$ $L = 0.650$ $\mu_0 = 0.9$
 STABLE FOR $a_1 < 1087$

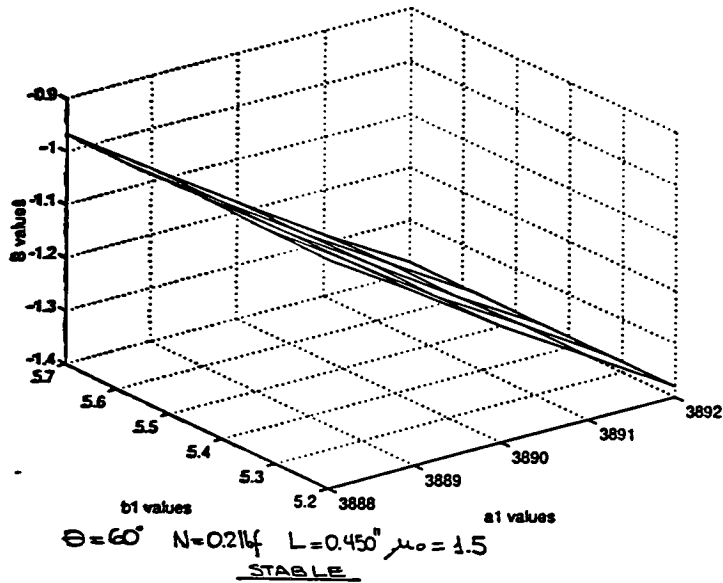
FIGURE 31

Stability Surface for Mathieu Equation



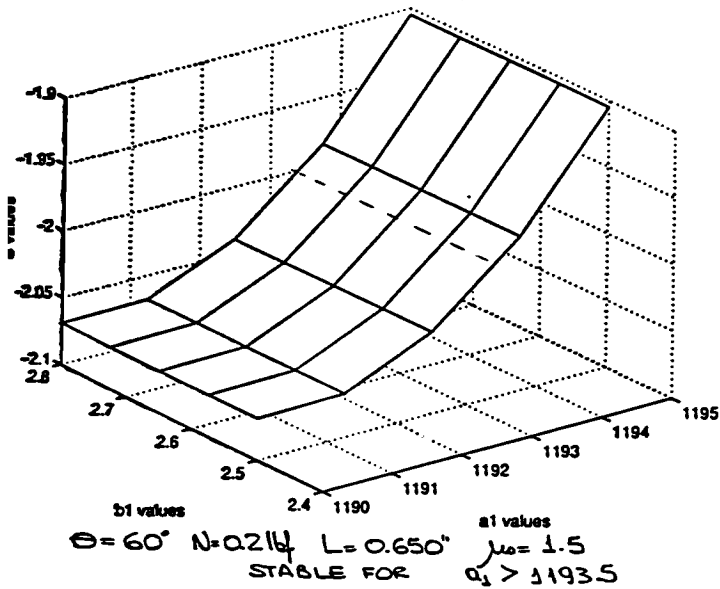
STABLE

Stability Surface for Mathieu Equation



STABLE

Stability Surface for Mathieu Equation



Contour Plot for $S=2$ or $S=2$

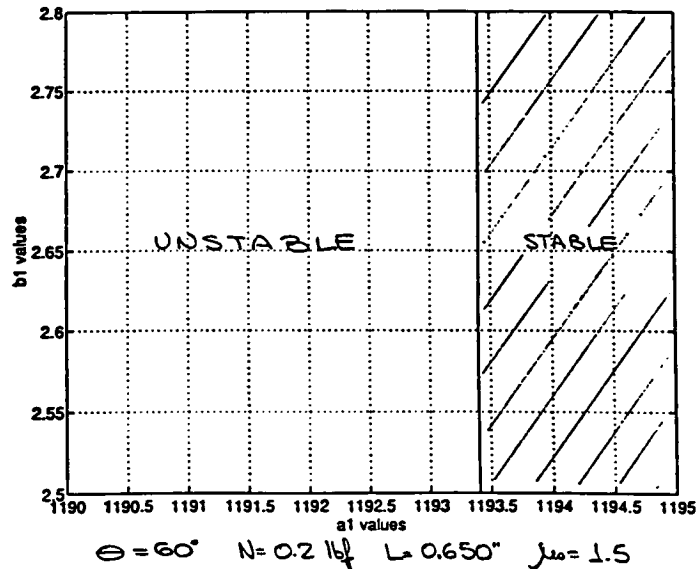


TABLE 2
TABLES OF MATHIEU COEFFICIENTS FOR VARIOUS
PARAMETERS IN THE CANTILEVER BEAM

$E=29 \times 10^6$ $\gamma=0.283$ THICKNESS=0.001" WIDTH=1.0" $\mu_1=0.01\mu_0$

$\theta = 30^\circ$ $N=0.1$ lbf

| L | μ_0 | | | |
|-------|---------|-------|--------|-------|
| | 0.3 | | 0.9 | |
| | a1 | b1 | a1 | b1 |
| 0.250 | 5283.1 | 21.99 | 57228 | 65.96 |
| 0.450 | 6404.6 | 6.79 | 7761.8 | 20.36 |
| 0.650 | 1966.5 | 3.25 | 2617.0 | 9.757 |

$\theta = 45^\circ$ $N=0.1$ lbf

| L | μ_0 | | | |
|-------|---------|-------|--------|-------|
| | 0.3 | | 0.9 | |
| | a1 | b1 | a1 | b1 |
| 0.250 | 54180 | 17.95 | 57770 | 53.85 |
| 0.450 | 6821.0 | 5.54 | 7929.1 | 16.62 |
| 0.650 | 2166.1 | 2.66 | 2697.2 | 7.97 |

$\theta = 60^\circ$ $N=0.1$ lbf

| L | μ_0 | | | |
|-------|---------|-------|--------|-------|
| | 0.3 | | 0.9 | |
| | a1 | b1 | a1 | b1 |
| 0.250 | 54999 | 12.69 | 57538 | 38.08 |
| 0.450 | 57073.8 | 3.92 | 7857.4 | 11.75 |
| 0.650 | 2287.3 | 1.88 | 2662.8 | 5.63 |

$\theta = 30^\circ$ $N=0.2$ lbf

| L | μ_0 | | | |
|-------|---------|-------|--------|--------|
| | 0.3 | | 0.9 | |
| | a1 | b1 | a1 | b1 |
| 0.250 | 59261 | 43.97 | 68055 | 131.91 |
| 0.450 | 8389.1 | 13.57 | 11103 | 40.71 |
| 0.650 | 2917.7 | 6.50 | 4218.6 | 19.51 |

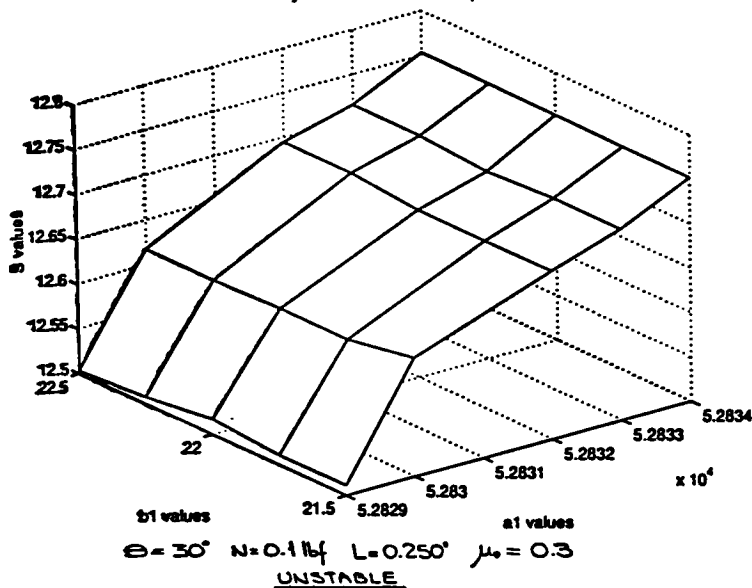
$\theta = 45^\circ$ $N=0.2$ lbf

| L | μ_0 | | | |
|-------|---------|-------|-------|--------|
| | 0.3 | | 0.9 | |
| | a1 | b1 | a1 | b1 |
| 0.250 | 61959 | 35.90 | 69139 | 107.71 |
| 0.450 | 9221.9 | 11.08 | 11438 | 33.24 |
| 0.650 | 3316.8 | 5.31 | 4379 | 15.93 |

$\theta = 60^\circ$ $N=0.2$ lbf

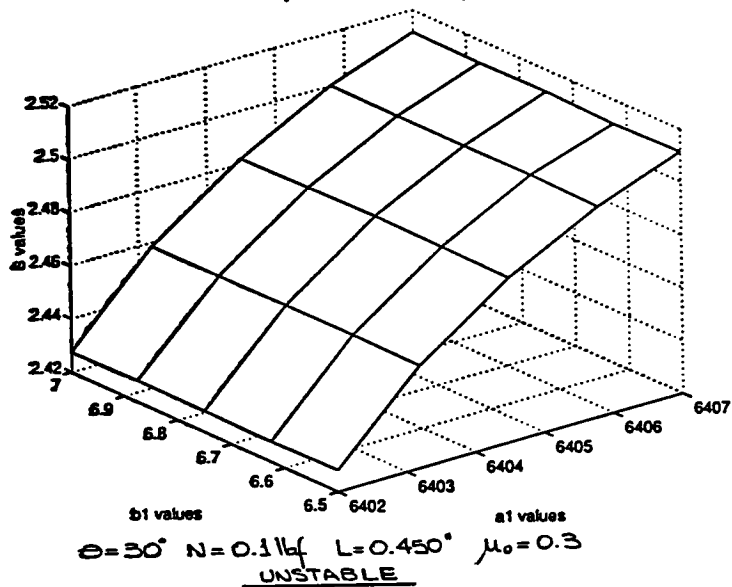
| L | μ_0 | | | |
|-------|---------|-------|--------|-------|
| | 0.3 | | 0.9 | |
| | a1 | b1 | a1 | b1 |
| 0.250 | 63597 | 25.39 | 68674 | 76.16 |
| 0.450 | 9727.5 | 7.84 | 11295 | 23.51 |
| 0.650 | 3559.1 | 3.76 | 4310.2 | 11.27 |

Stability Surface for Mathieu Equation



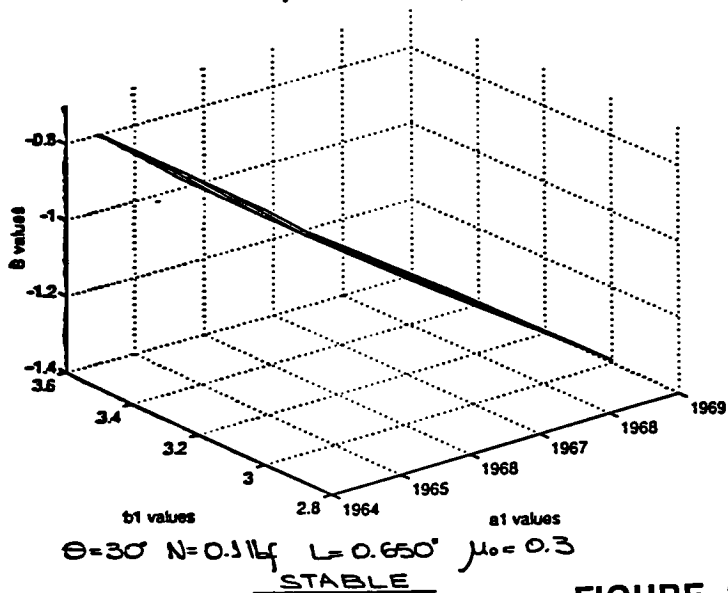
UNSTABLE

Stability Surface for Mathieu Equation



UNSTABLE

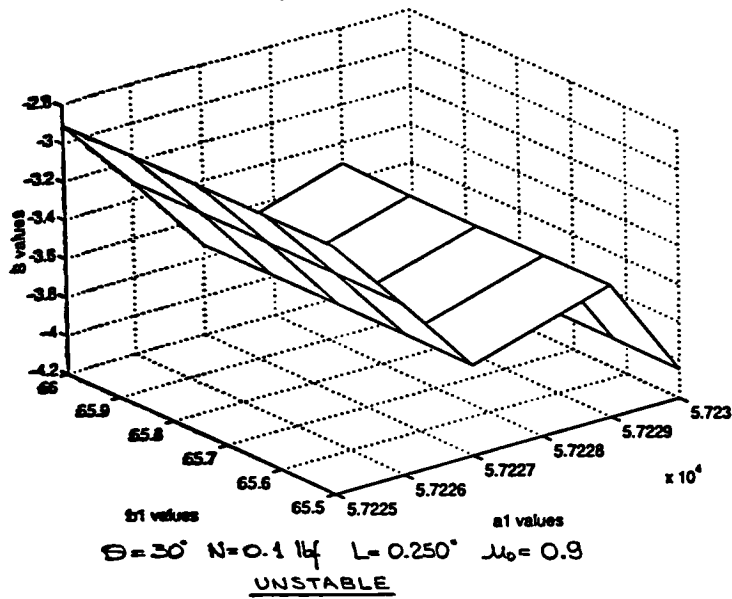
Stability Surface for Mathieu Equation



UNSTABLE

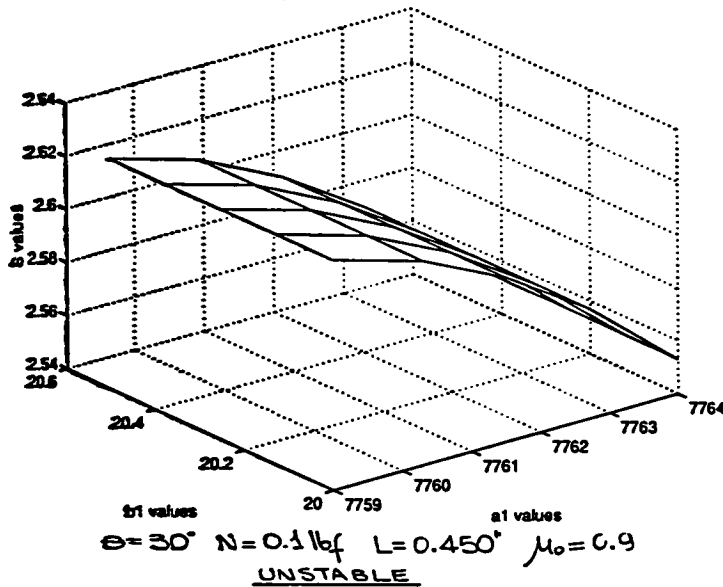
FIGURE 33

Stability Surface for Mathieu Equation



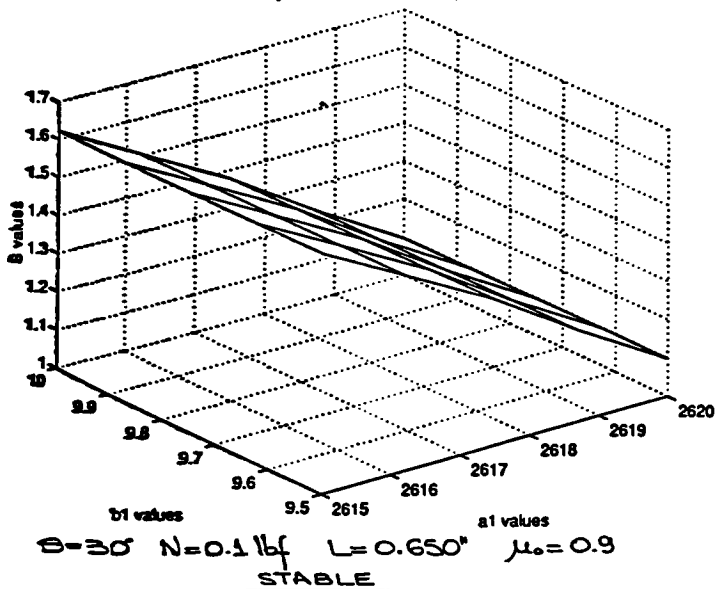
UNSTABLE

Stability Surface for Mathieu Equation



UNSTABLE

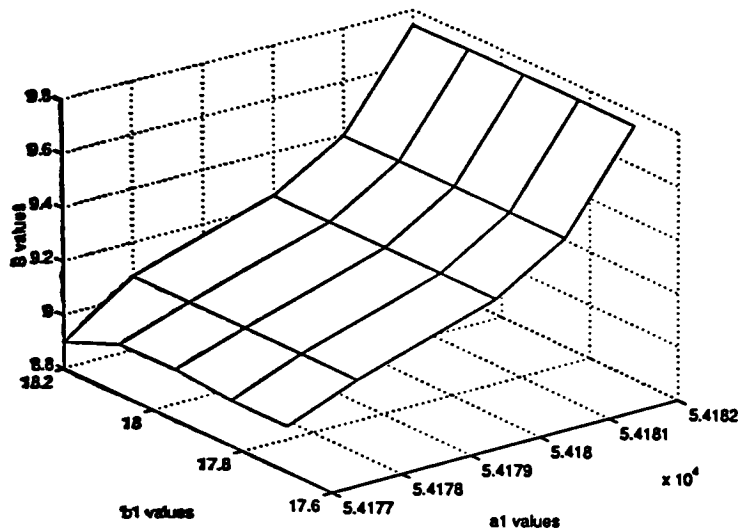
Stability Surface for Mathieu Equation



STABLE

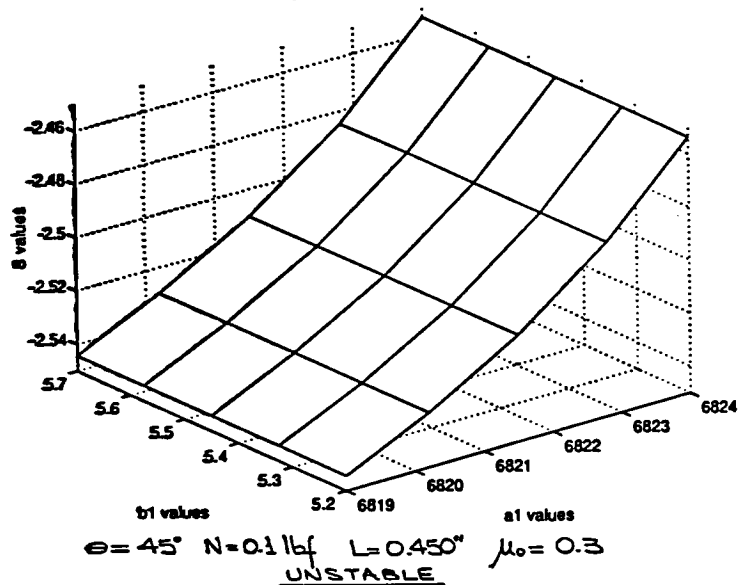
FIGURE 34

Stability Surface for Mathieu Equation



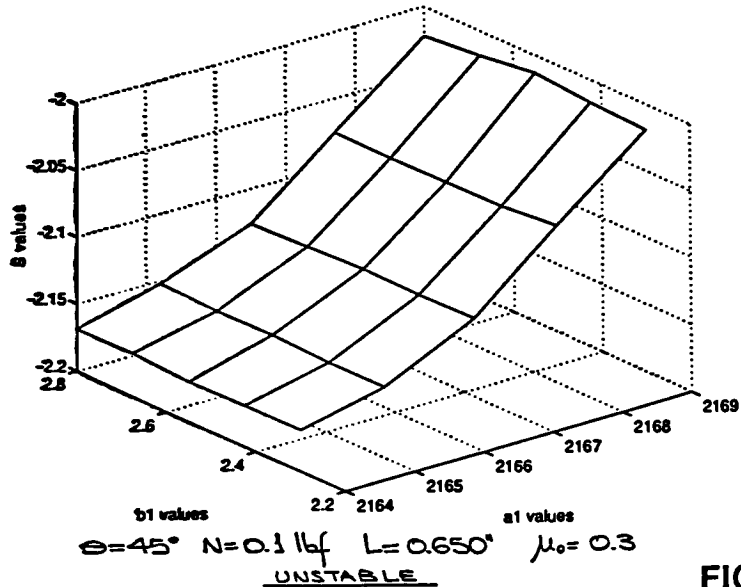
UNSTABLE

Stability Surface for Mathieu Equation



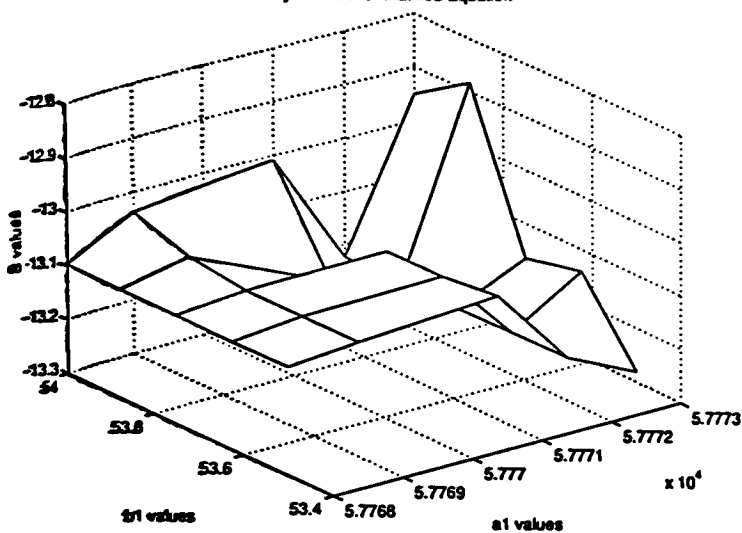
UNSTABLE

Stability Surface for Mathieu Equation



UNSTABLE

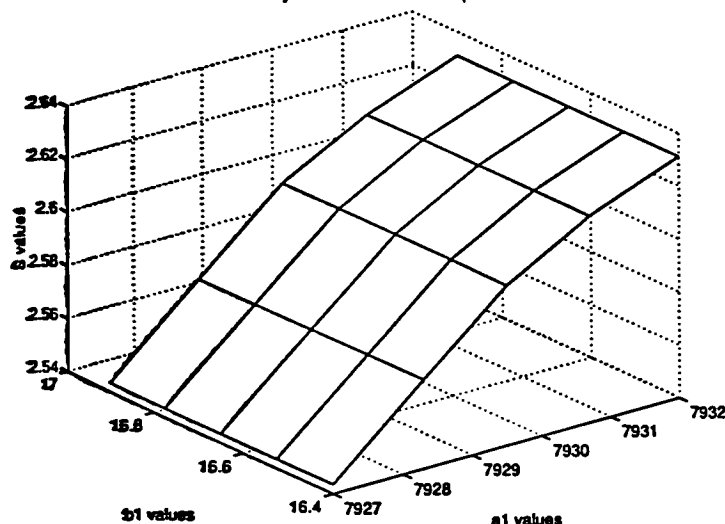
Stability Surface for Mathieu Equation



$\theta = 45^\circ$ $N = 0.116$ $L = 0.250^\circ$ $\mu_0 = 0.9$
UNSTABLE

UNSTABLE

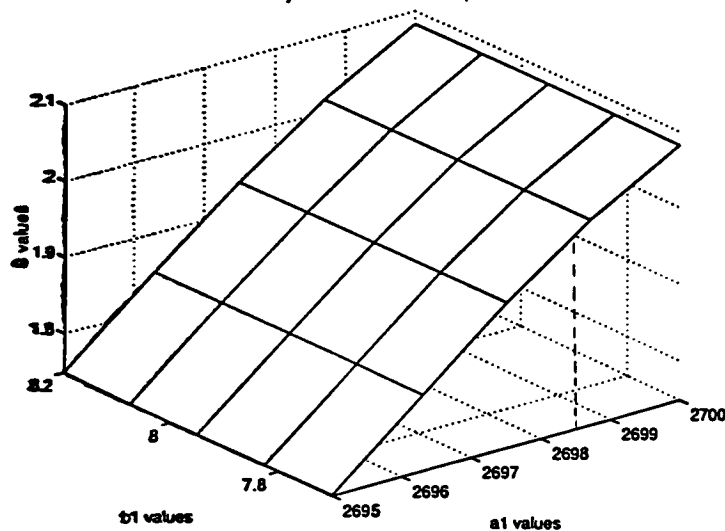
Stability Surface for Mathieu Equation



$\theta = 45^\circ$ $N = 0.116$ $L = 0.450^\circ$ $\mu_0 = 0.9$
UNSTABLE

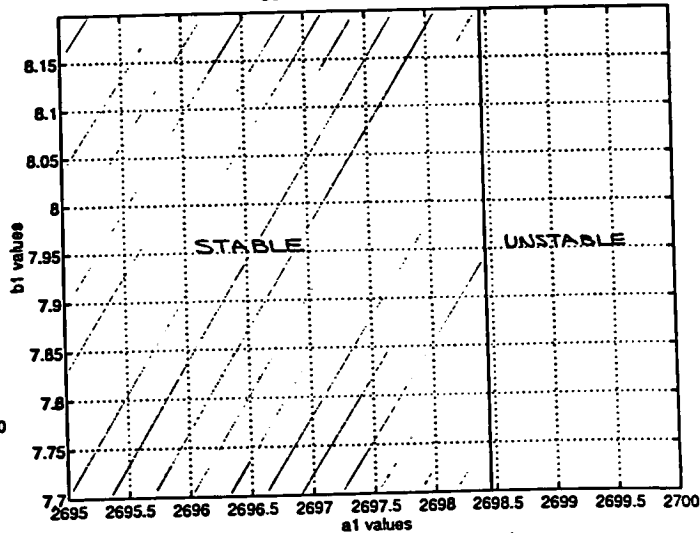
UNSTABLE

Stability Surface for Mathieu Equation



$\theta = 45^\circ$ $N = 0.116$ $L = 0.650^\circ$ $\mu_0 = 0.9$
STABLE FOR $2695.0 < 2698.5$

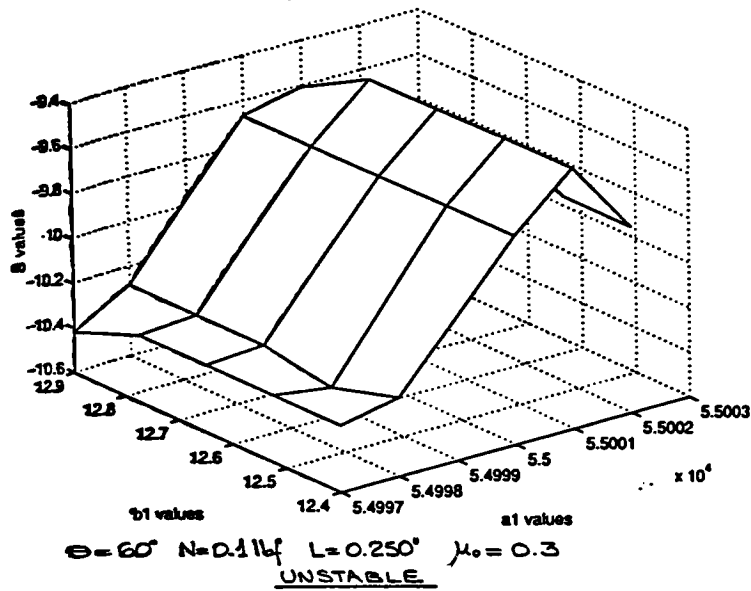
Contour Plot for $S=-2$ or $S=2$



$\theta = 45^\circ$ $N = 0.116$ $L = 0.650^\circ$ $\mu_0 = 0.9$

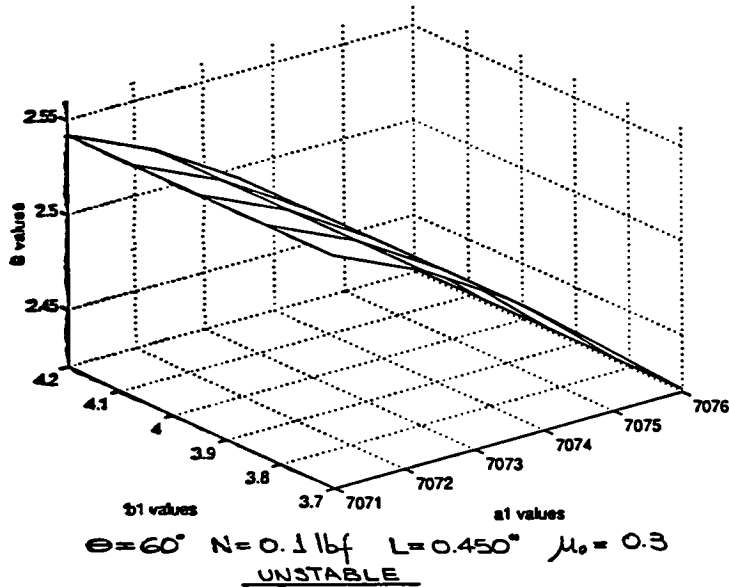
FIGURE 36

Stability Surface for Mathieu Equation



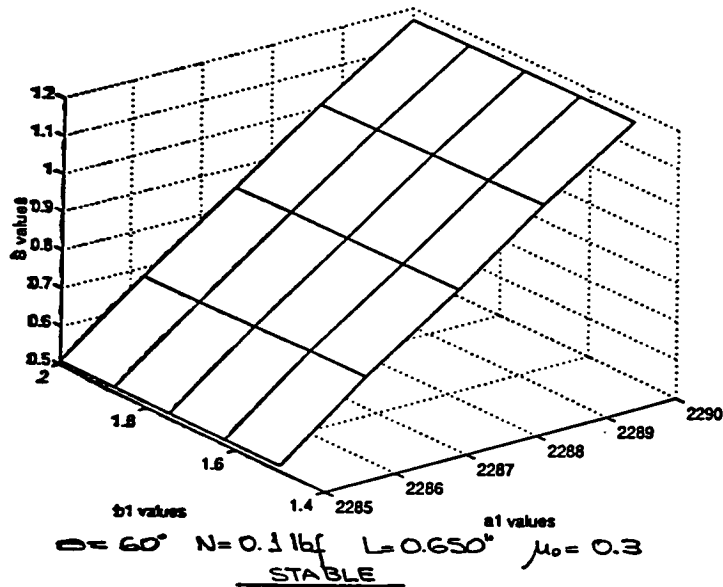
UNSTABLE

Stability Surface for Mathieu Equation



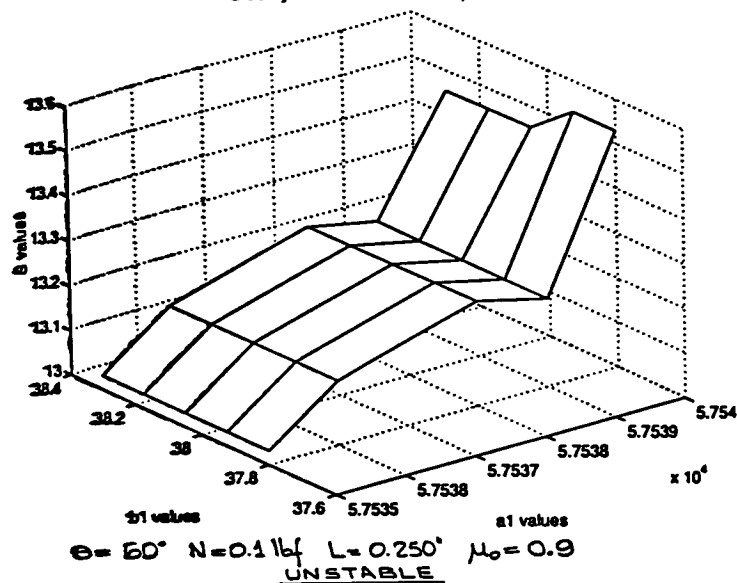
UNSTABLE

Stability Surface for Mathieu Equation



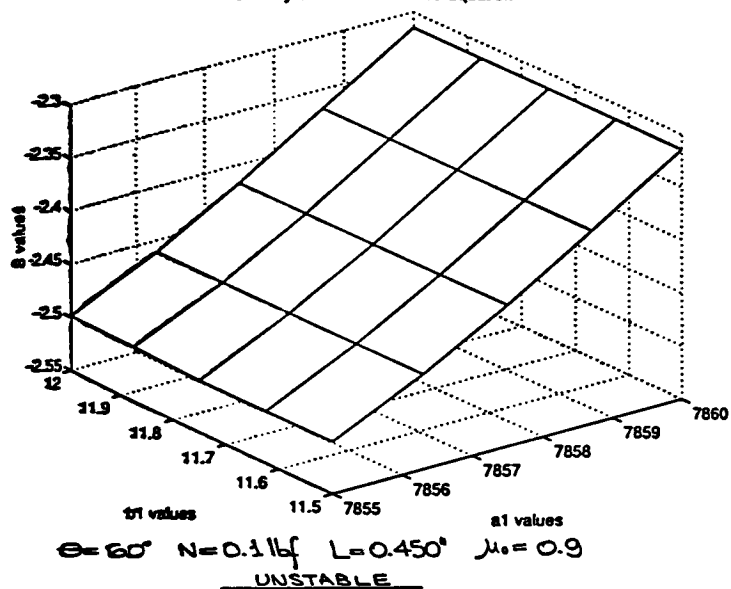
STABLE

Stability Surface for Mathieu Equation



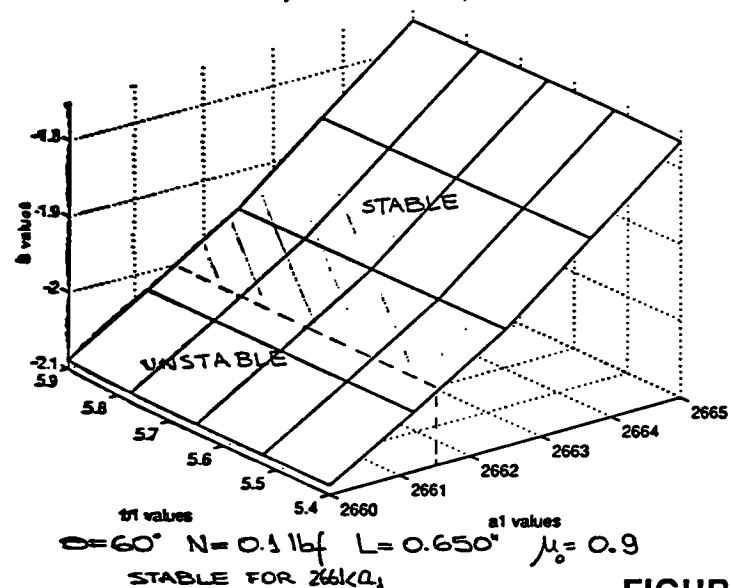
UNSTABLE

Stability Surface for Mathieu Equation



UNSTABLE

Stability Surface for Mathieu Equation



Contour Plot for $S=-2$ or $S=2$

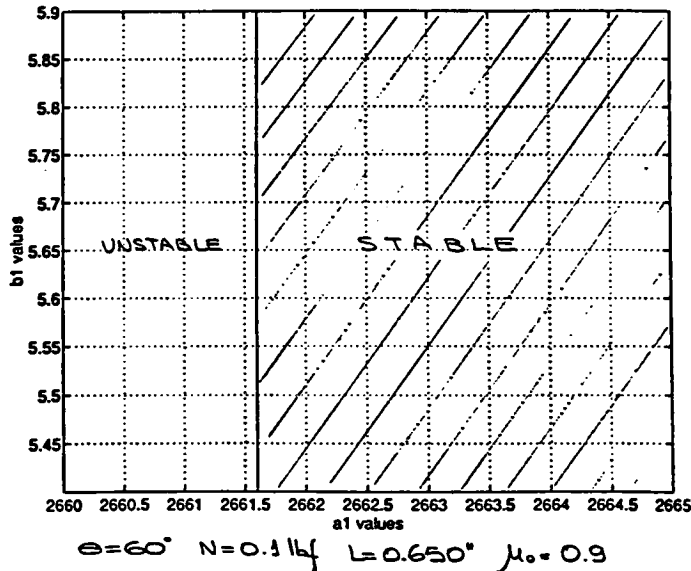
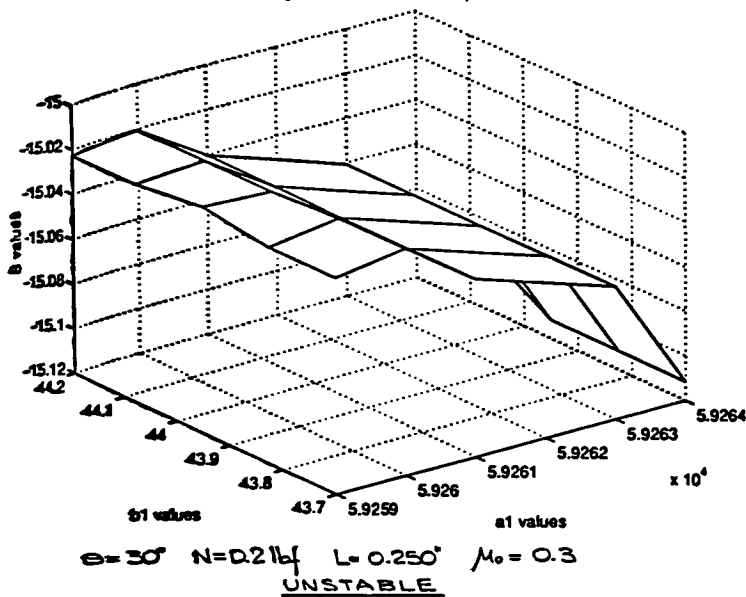


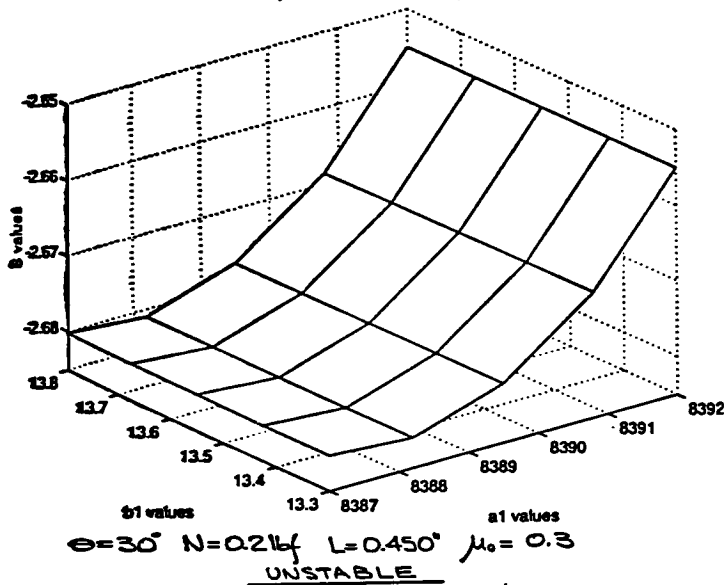
FIGURE 38

Stability Surface for Mathieu Equation



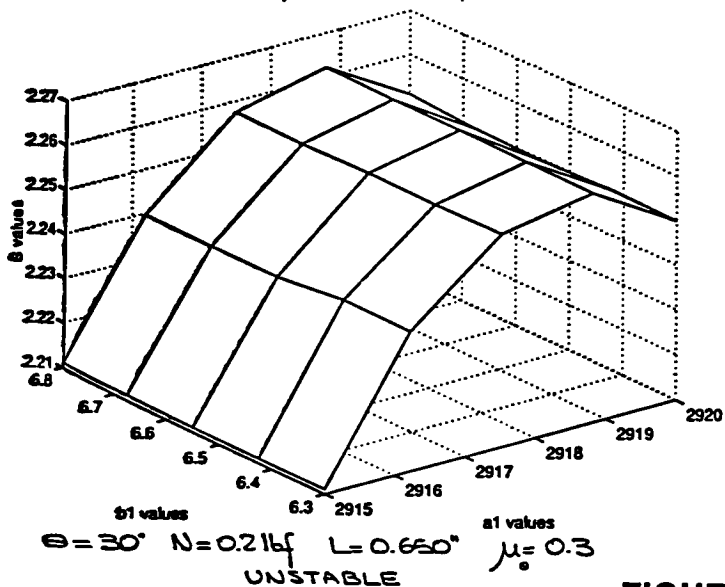
UNSTABLE

Stability Surface for Mathieu Equation



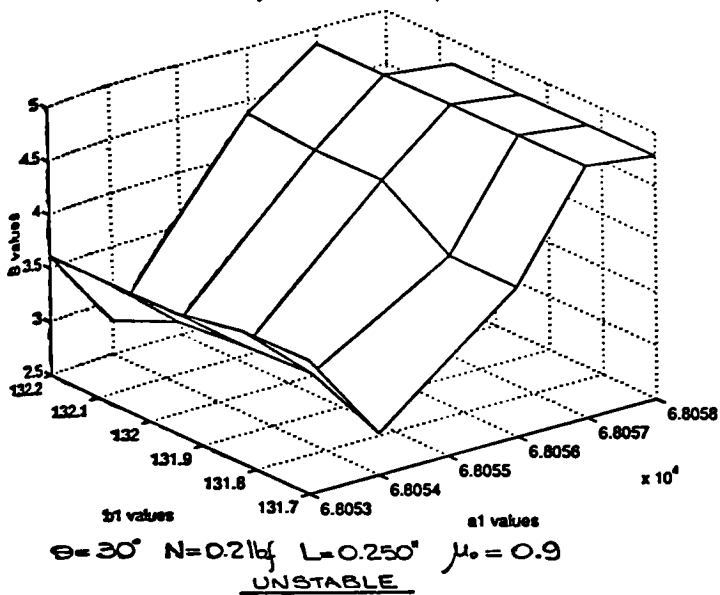
UNSTABLE

Stability Surface for Mathieu Equation



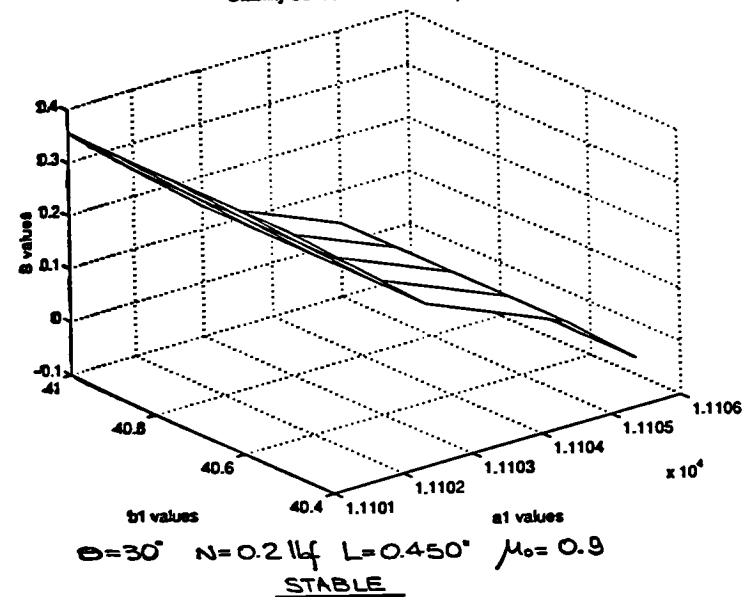
UNSTABLE

Stability Surface for Mathieu Equation



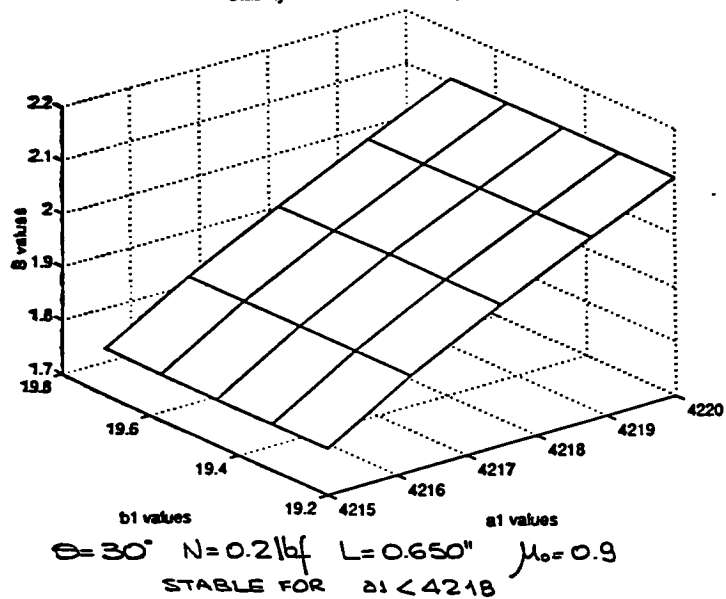
UNSTABLE

Stability Surface for Mathieu Equation

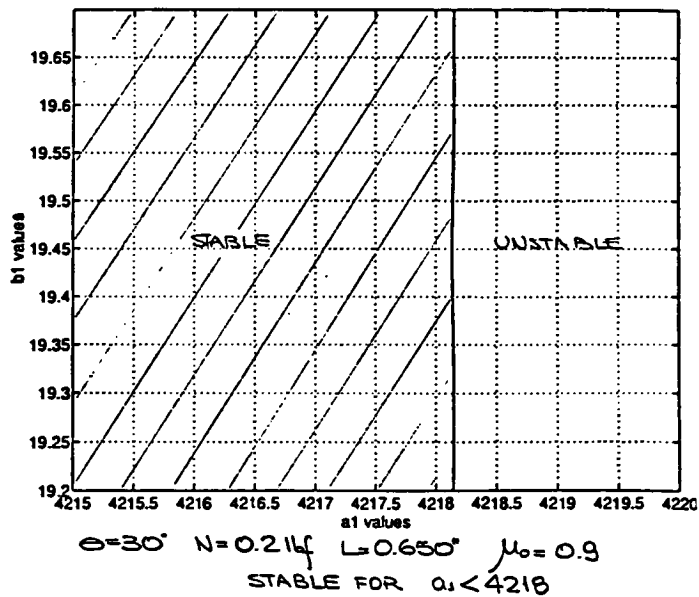


STABLE

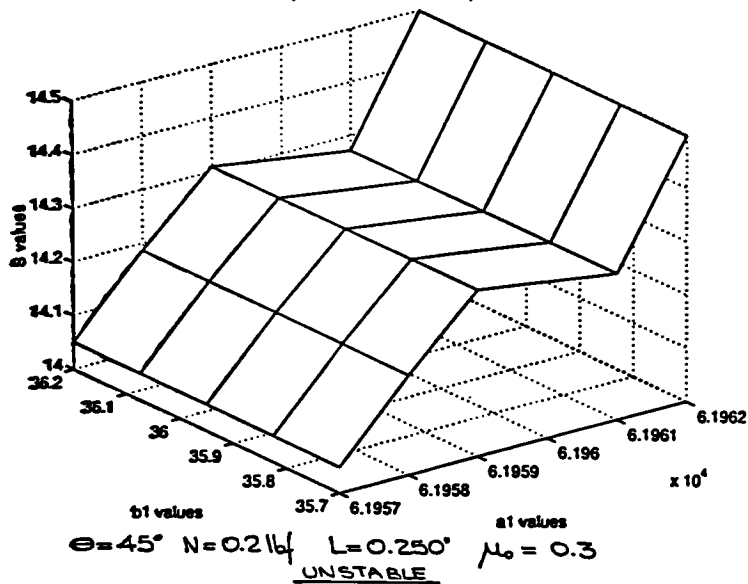
Stability Surface for Mathieu Equation



Contour Plot for $S=-2$ or $S=2$

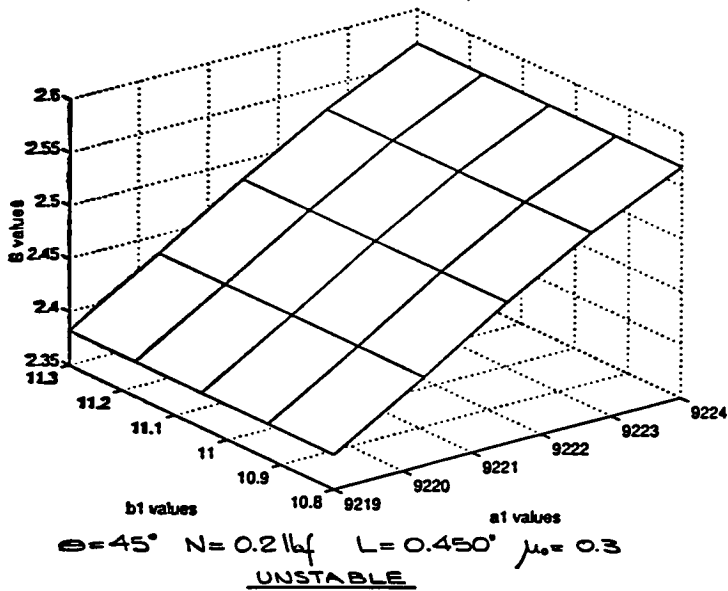


Stability Surface for Mathieu Equation



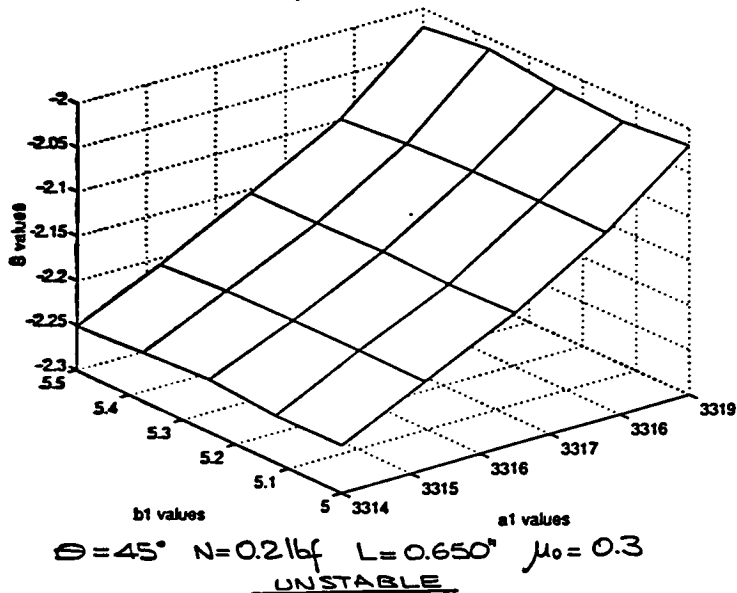
UNSTABLE

Stability Surface for Mathieu Equation



UNSTABLE

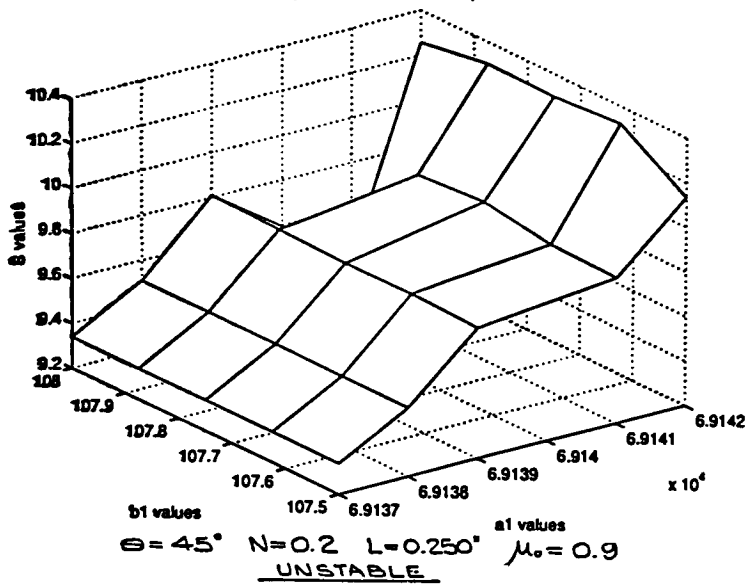
Stability Surface for Mathieu Equation



UNSTABLE

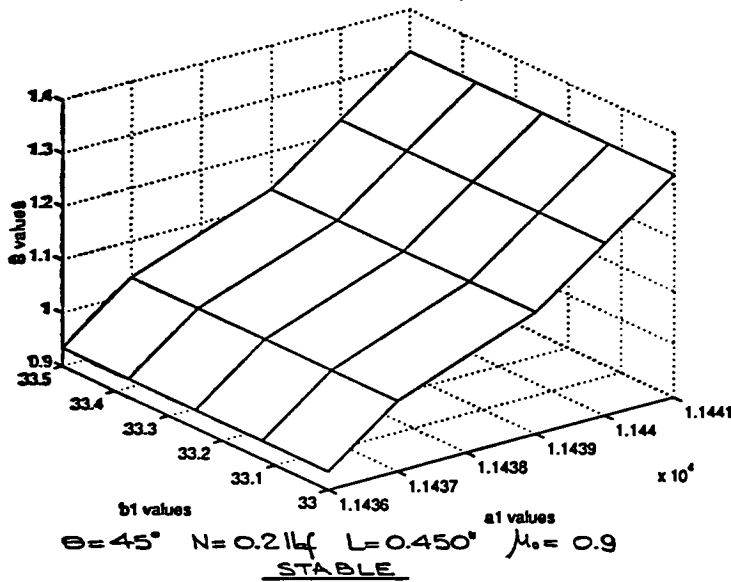
FIGURE 41

Stability Surface for Mathieu Equation



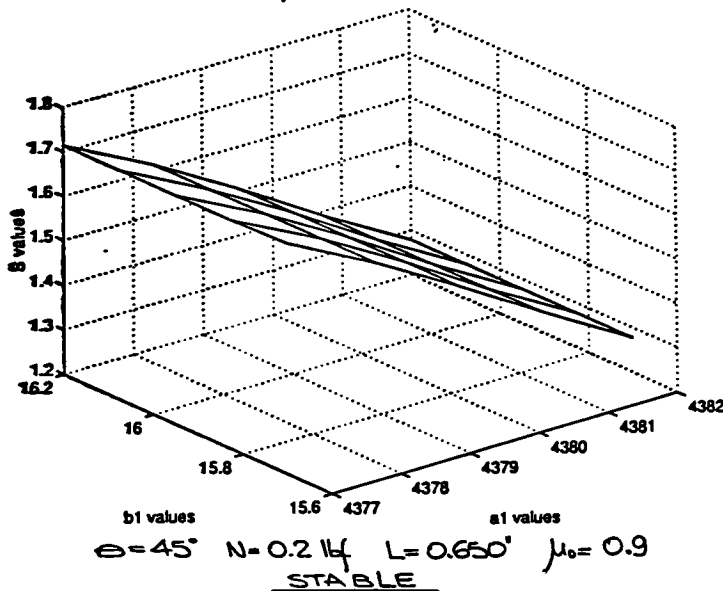
UNSTABLE

Stability Surface for Mathieu Equation



STABLE

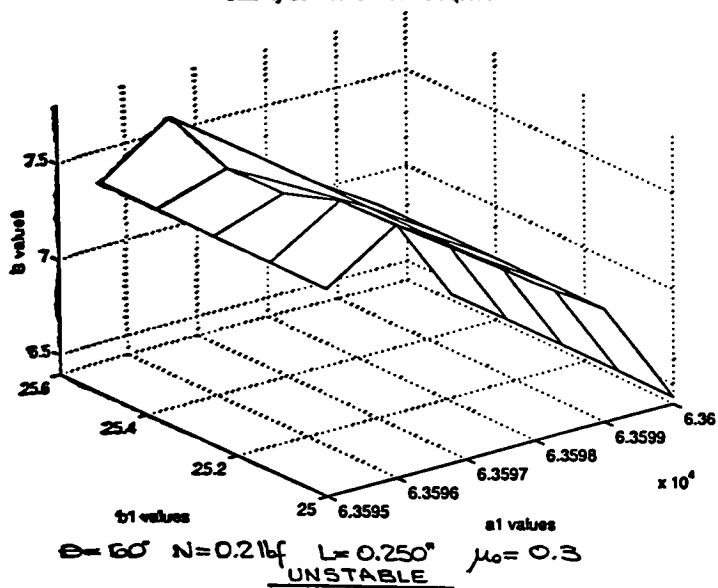
Stability Surface for Mathieu Equation



STABLE

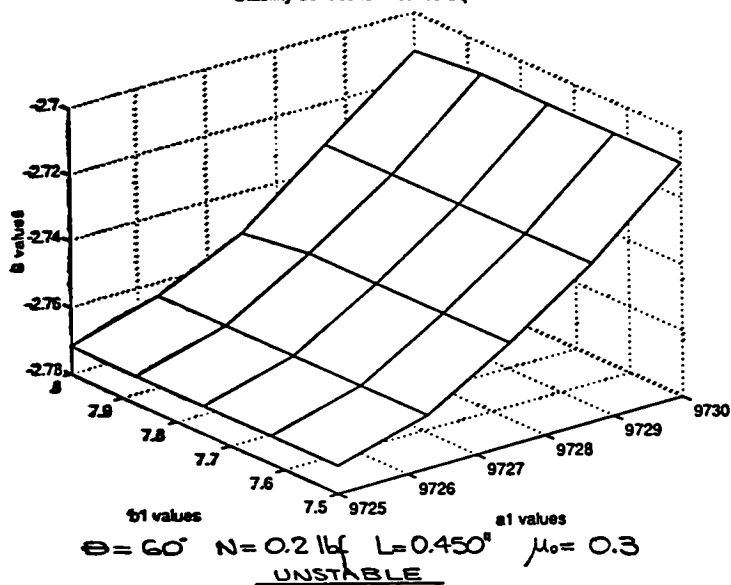
FIGURE 42

Stability Surface for Mathieu Equation



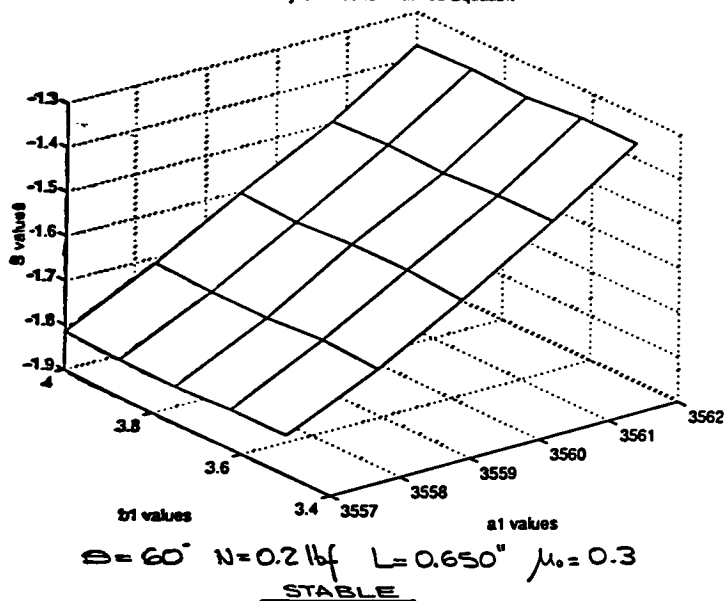
UNSTABLE

Stability Surface for Mathieu Equation



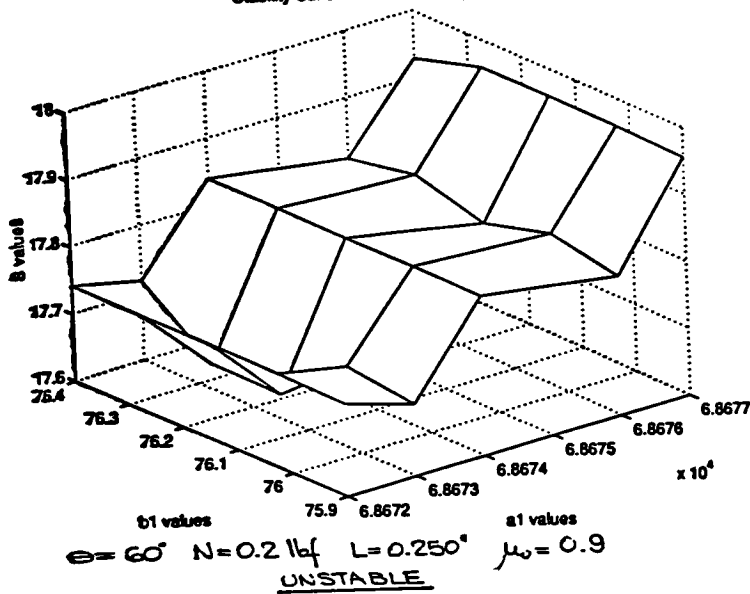
UNSTABLE

Stability Surface for Mathieu Equation



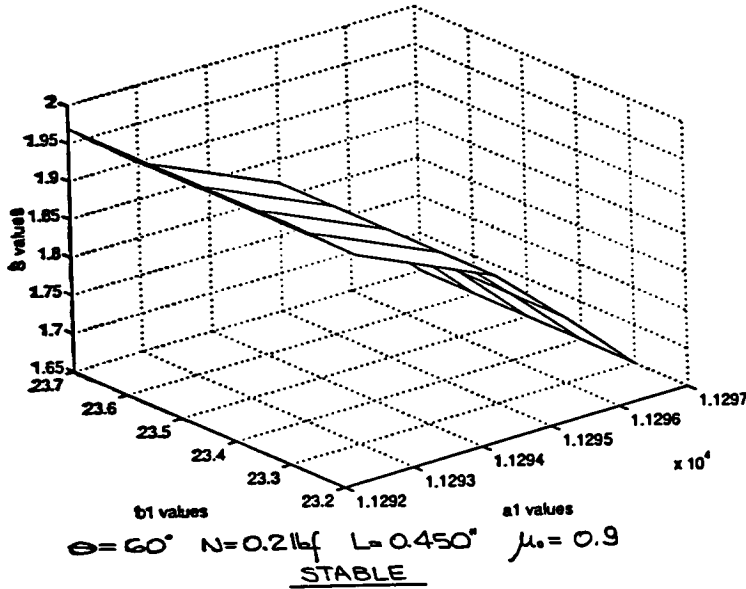
STABLE

Stability Surface for Mathieu Equation



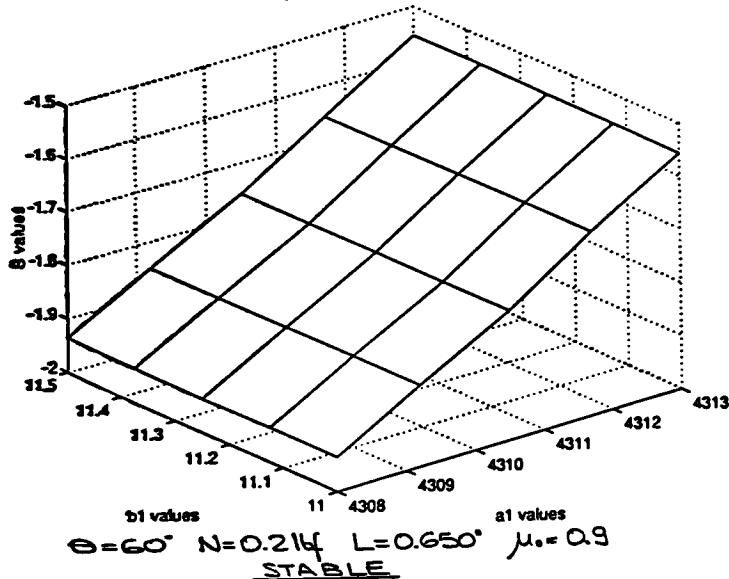
UNSTABLE

Stability Surface for Mathieu Equation



STABLE

Stability Surface for Mathieu Equation



STABLE

FIGURE 44

[5] Conclusions

In the thesis the stability of a cantilevered beam has been studied by using the energy method of writing the equations of motion for the forced beam. Application of Hamilton's principle yields the geometric and natural boundary conditions. The resulting equations of motion can not be solved in closed form and Galerkin's method is applied using an approximate or admissible function for the displacement function of the beam. Galerkin's method results into a differential equation of time with a periodically varying coefficient of the first order term. This time differential equation has a form corresponding to a Mathieu's equation that has been extensively studied. In particular, a stability study of the response has been presented in the form of a plot called Strutt's diagram. Strutt's diagram is a contour plot that defines stable and unstable regions depending on the values of the coefficients in Mathieu's equation. This plot has been used in this paper as a guide in determining stability conditions for our case study. For values of the coefficients in Mathieu's equation not covered by the plot, one has to use a numerical method such as Runge-Kutta to evaluate the values of S and verify that the beam falls within the range required for stability as presented above.

In practice, at higher friction, the beam tends to more readily become unstable, though. Furthermore, an increase in the angle θ above 30° combined with higher friction also leads in unstable behavior. These two combinations for unstable behavior are not properly explained by the present analysis. One possible reason may be that we did not model the free end of the beam properly, that is, the free end was not properly constrained, or that excentricities in the roller will lead to variability in the normal load N which is not modelled here. Additionally, web velocity

fluctuations can occur at much higher frequencies than once per revolution of the roller. A higher value of ω would drive the values of coefficient a_1 to much smaller values and it would thus position the beam in regions of instability. Slip-stick motion is greatly reduced in practice via lubrication.

In applications of the beam in contact with a harder roller, it is desirable to reduce the beam stiffness to achieve lower variability in the normal load due to excentricity in the roller.

The analysis presented in this thesis need further experimental and analytical verification and the beam constraints might also need further refinement. The effect of slip velocity and the adhesion of polyurethane beam tip in contact with the roller on the stability behavior needs to be investigated since they may play a more important role in explaining the disagreements in behavior between the model and in practice.

The author would like to thank Dr. Joseph Torok for the guidance, support provided in the literature search, for analytical support in the area of stability of beams, bifurcation analysis of static and dynamic problems, MAPLE, MATLAB, Galerkin's method and in the development of the equations of motion via Hamilton's equation, and for moral support during the assembly of the thesis and its defense. The author is immensely grateful for the Mechanical Engineering department committee members for participating in his defense and providing some guidance and proof-reading. In particular, the author would like to thank Dr. Hetnarski for the meticulous and precise input provided to make some points more clear to the reader.

APPENDIX A

Derivation of Strain Along The Neutral Axis Of A Deformed Cantilever Beam

A cantilever beam in both the undeformed and deformed shape is shown in figure A-1(a). An infinitesimally small element of length dx undergoes bending and stretching which results in its new location and deformed length ds . The strain along the neutral axis can be expressed as:

$$\epsilon = (ds - dx)/dx \quad (A-1)$$

Figure A-1(b) magnifies the elements dx and ds and shows their corresponding deformations. From this figure, we can estimate ds as:

$$ds^2 = a^2 + dv^2 \quad (A-2)$$

where

$$a = du + dx \quad (A-3)$$

Substituting equation (A-3) into equation (A-2), we have:

$$ds^2 = (du + dx)^2 + dv^2 \quad (A-4)$$

or

$$ds = [(du + dx)^2 + dv^2]^{0.5} \quad (A-5)$$

Substituting equation (A-5) into equation (A-1) and simplifying:

$$\epsilon = \{ [(du + dx)^2 + dv^2]^{0.5} - dx \} / dx \quad (A-6)$$

or

$$\varepsilon = \{ [(u' + 1)^2 + (v')^2]^{0.5} - 1 \} \quad (A-7)$$

The first term in equation (A-7) can be factorized as:

$$[(u' + 1)^2 + (v')^2]^{0.5} = (u' + 1) * [1 + (v')^2 / (u' + 1)^2]^{0.5} \quad (A-8)$$

By the binomial series expansion, we can write that:

$$[1 + x]^{0.5} = 1 + 0.5 * x - [0.5 * (1/4)] * x^2 \dots$$

Using the binomial series expansion on the second factor of equation (A-8) and taking only the first two terms, we get:

$$[1 + (v')^2 / (u' + 1)^2]^{0.5} = 1 + 0.5 * [(v')^2 / (u' + 1)^2] \quad (A-9)$$

Substituting equation (A-9) into equation (A-8), we get:

$$[(u' + 1)^2 + (v')^2]^{0.5} = (u' + 1) * \{ 1 + 0.5 * [(v')^2 / (u' + 1)^2] \} \quad (A-10)$$

Expanding the above expression results in:

$$[(u' + 1)^2 + (v')^2]^{0.5} = 1 + u' + 0.5 * [(v')^2 / (u' + 1)] \quad (A-11)$$

Again by the binomial expansion, we can write:

$$1 / (1 + u') = 1 - u' + (u')^2 \dots \text{ for } -1 < u' < 1 \quad (A-12)$$

Taking as an approximation the first two terms of the above expression and substituting back into equation (A-11):

$$[(u' + 1)^2 + (v')^2]^{0.5} = 1 + u' + 0.5*(1 - u')*(v')^2 \quad (\text{A-13})$$

Expanding equation (A-13):

$$[(u' + 1)^2 + (v')^2]^{0.5} = 1 + u' + 0.5*(v')^2 - 0.5*u'*(v')^2 \quad (\text{A-14})$$

Neglecting terms higher than second order, we obtain:

$$[(u' + 1)^2 + (v')^2]^{0.5} = 1 + u' + 0.5*(v')^2 \quad (\text{A-15})$$

Finally, substituting equation (A-15) back into equation (A-7):

$$\varepsilon = u' + 0.5*(v')^2 \quad (\text{A-16})$$

STRAIN ALONG THE NEUTRAL AXIS

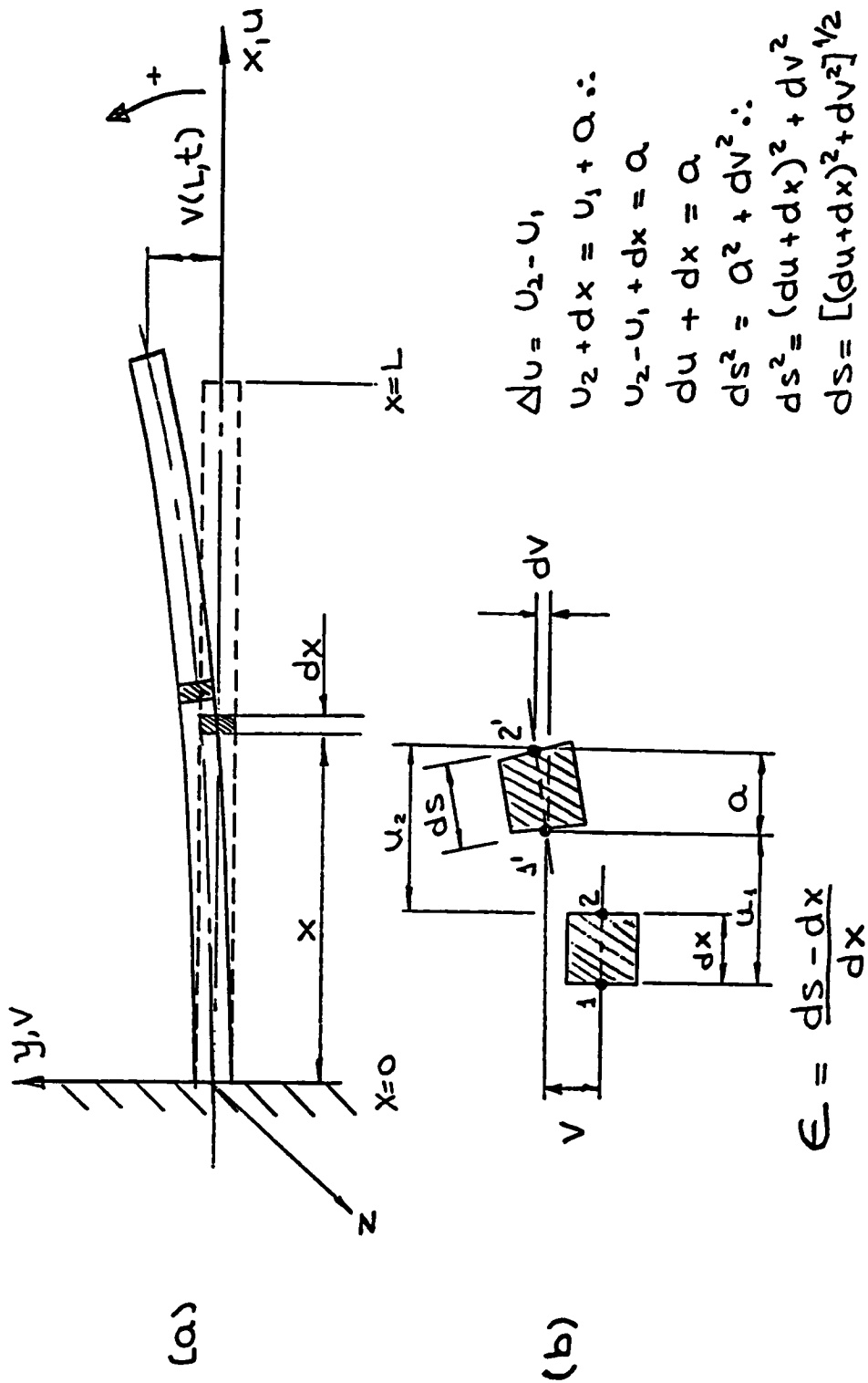


FIGURE A-1

APPENDIX B

FREE VIBRATION OF A CANTILEVER BEAM

For the cantilever beam shown in figure A-1, let $u(x,t)$ and $v(x,t)$ be the axial and transverse displacements of the beam at a distance x from the fixed end and at time t . The equations of motion for such a beam can be derived via the energy method by utilizing Hamilton's principle. Hamilton's principle is represented by the equation:

$$\int_{t_1}^{t_2} \delta(T - V) dt + \int_{t_1}^{t_2} \delta W_{nc} dt = 0 \quad (B-1)$$

where

T = total kinetic energy of the system

V = potential energy of the system, including the strain energy

δW_{nc} = virtual work done by nonconservative forces, including damping forces and external forces not accounted for in V .

t_1, t_2 = times at which the configuration of the system is known.

For the free beam, the virtual work is zero since there is no force in action (except for internal damping which is neglected here). The kinetic energy of the beam may be described by:

$$T = 0.5 \rho \int_0^L [(u_t)^2 + (v_t)^2] dx \quad (B-2)$$

The potential energy here consists of the strain energy due to bending and that due to tension or compression, that is:

$$V = 0.5 \int_0^L [E \cdot I \cdot (v'')^2] dx + 0.5 \int_0^L [E \cdot A \cdot (u' + 0.5 \cdot v'^2)^2] dx \quad (B-3)$$

Applying the variational operator in the above expressions, we get:

$$\delta T = \rho \int_0^L [(u_t) \cdot (\delta u_t) + (v_t) \cdot (\delta v_t)] dx \quad (B-4)$$

and

$$\delta V = \int_0^L [E \cdot I \cdot (v'') \cdot (\delta v'')] dx + \int_0^L [E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot \delta(u' + 0.5 \cdot v'^2)] dx \quad (B-5)$$

Integrating δV by part twice, we have:

$$\begin{aligned} \delta V = & E \cdot I \cdot (v'') \cdot (\delta v') \Big|_0^L - E \cdot I \cdot (v''') \cdot \delta v \Big|_0^L + \int_0^L [E \cdot I \cdot (v''''') \cdot \delta v] dx \\ & + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot \delta u \Big|_0^L - \int_0^L [E \cdot A \cdot (u' + 0.5 \cdot v'^2)' \cdot \delta u] dx \\ & + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' \cdot \delta v \Big|_0^L - \int_0^L \{ E \cdot A \cdot [(u' + 0.5 \cdot v'^2) \cdot v']' \cdot \delta v \} dx \quad (B-6) \end{aligned}$$

Substituting the above terms into Hamilton's equation:

$$\begin{aligned} \delta(T - V) = & \int_{t_1}^{t_2} \left\{ \rho \int_0^L [(u_t) \cdot (\delta u_t) + (v_t) \cdot (\delta v_t)] dx \right\} dt \\ & - \int_{t_1}^{t_2} \{ E \cdot I \cdot (v'') \cdot (\delta v') \Big|_0^L - E \cdot I \cdot (v''') \cdot \delta v \Big|_0^L + \int_0^L [E \cdot I \cdot (v''''') \cdot \delta v] dx \end{aligned}$$

$$\begin{aligned}
& + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot \delta u \Big|_0^L - \int_0^L [E \cdot A \cdot (u' + 0.5 \cdot v'^2)' \cdot \delta u] \cdot dx \\
& + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' \cdot \delta v \Big|_0^L - \int_0^L \{ E \cdot A \cdot [(u' + 0.5 \cdot v'^2) \cdot v']' \cdot \delta v \} \cdot dx \} \cdot dt = 0 \quad (B-7)
\end{aligned}$$

The first term in the above expression can be further simplified by manipulating the integrals and then integrating by parts:

$$\begin{aligned}
\int_{t_1}^{t_2} \delta T \cdot dt &= \int_{t_1}^{t_2} \left\{ \rho \cdot \int_0^L [u_t \cdot \delta u_t + v_t \cdot \delta v_t] \cdot dx \right\} \cdot dt = \\
& \int_0^L [\rho \cdot u_t \cdot \delta u|_{t=t_2} - \rho \cdot u_t \cdot \delta u|_{t=t_1}] \cdot dx - \int_{t_1}^{t_2} \int_0^L \rho \cdot u_{tt} \cdot \delta u \cdot dx \cdot dt \\
& + \int_0^L [\rho \cdot v_t \cdot \delta v|_{t=t_2} - \rho \cdot v_t \cdot \delta v|_{t=t_1}] \cdot dx - \int_{t_1}^{t_2} \int_0^L \rho \cdot v_{tt} \cdot \delta v \cdot dx \cdot dt \quad (B-8)
\end{aligned}$$

Substituting the above expression back into Hamilton's equation, we have:

$$\begin{aligned}
& \int_0^L [\rho^* u_t^* \delta u|_{t=t_2} - \rho^* u_t^* \delta u|_{t=t_1}]^* dx - \int_{t_1}^{t_2} \int_0^L \rho^* u_{tt}^* \delta u^* dx^* dt \\
& + \int_0^L [\rho^* v_t^* \delta v|_{t=t_2} - \rho^* v_t^* \delta v|_{t=t_1}]^* dx - \int_{t_1}^{t_2} \int_0^L \rho^* v_{tt}^* \delta v^* dx^* dt \\
& - \int_{t_1}^{t_2} \{ E^* I^* (v'')^* \delta v|_0^L - E^* I^* (v''')^* \delta v|_0^L + \int_0^L [E^* I^* (v''')^* \delta v]^* dx \\
& + E^* A^* (u' + 0.5^* v'^2)^* \delta u|_0^L - \int_0^L [E^* A^* (u' + 0.5^* v'^2)^* \delta u]^* dx \\
& + E^* A^* (u' + 0.5^* v'^2)^* v'^* \delta v|_0^L - \int_0^L \{ E^* A^* [(u' + 0.5^* v'^2)^* v']^* \delta v \}^* dx \}^* dt \\
& = 0
\end{aligned} \tag{B-9}$$

Separating similar terms in the above expression, we have:

$$\begin{aligned}
& \int_0^L [\rho^* u_t^* \delta u|_{t=t_2} - \rho^* u_t^* \delta u|_{t=t_1}]^* dx + \int_0^L [\rho^* v_t^* \delta v|_{t=t_2} - \rho^* v_t^* \delta v|_{t=t_1}]^* dx \\
& - \int_{t_1}^{t_2} \int_0^L \{ \rho^* u_{tt} - E^* A^* (u' + 0.5^* v'^2)' \}^* \delta u^* dx^* dt \\
& - \int_{t_1}^{t_2} \int_0^L \{ \rho^* v_{tt} + E^* I^* v'''' - E^* A^* [(u' + 0.5^* v'^2)^* v']' \}^* \delta v^* dx^* dt \\
& - \int_{t_1}^{t_2} E^* I^* v'''^* \delta v'|_{x=L}^* dt - \int_{t_1}^{t_2} E^* I^* v''^* \delta v'|_{x=0}^* dt
\end{aligned}$$

$$- \int_{t_1}^{t_2} \{ - E \cdot I \cdot v'''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' \} \cdot \delta v|_{x=L} \cdot dt$$

$$- \int_{t_1}^{t_2} \{ - E \cdot I \cdot v'''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' \} \cdot \delta v|_{x=0} \cdot dt$$

$$- \int_{t_1}^{t_2} \{ E \cdot A \cdot (u' + 0.5 \cdot v'^2) \} \cdot \delta u|_{x=L} \cdot dt - \int_{t_1}^{t_2} \{ E \cdot A \cdot (u' + 0.5 \cdot v'^2) \} \cdot \delta u|_{x=0} \cdot dt = 0 \quad (B-10)$$

For the above equation to be satisfied, the following conditions must be held :

$$\rho \cdot u_t \cdot \delta u|_{t=t_2} = 0, \quad (B-10i)$$

$$\rho \cdot u_t \cdot \delta u|_{t=t_1} = 0, \quad (B-10ii)$$

$$\rho \cdot v_t \cdot \delta v|_{t=t_2} = 0, \quad (B-10iii)$$

$$\rho \cdot v_t \cdot \delta v|_{t=t_1} = 0, \quad (B-10iv)$$

$$\{ \rho \cdot u_{tt} - E \cdot A \cdot (u' + 0.5 \cdot v'^2)' \} \cdot \delta u = 0 \quad (B-10v)$$

$$\{ \rho \cdot v_{tt} + E \cdot I \cdot v'''' - E \cdot A \cdot [(u' + 0.5 \cdot v'^2) \cdot v']' \} \cdot \delta v = 0 \quad (B-10vi)$$

$$\{ E \cdot I \cdot v'' \} \cdot \delta v'|_{x=L} = 0 \quad (B-10vii)$$

$$\{ E \cdot I \cdot v'' \} \cdot \delta v'|_{x=0} = 0 \quad (B-10viii)$$

$$\{- E \cdot I \cdot v''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' \} \cdot \delta v \big|_{x=L} = 0 \quad (\text{B-10ix})$$

$$\{- E \cdot I \cdot v''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' \} \cdot \delta v \big|_{x=0} = 0 \quad (\text{B-10x})$$

$$E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot \delta u \big|_{x=L} = 0 \quad (\text{B-10xi})$$

$$E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot \delta u \big|_{x=0} = 0 \quad (\text{B-10xii})$$

In using Hamilton's principle, it is assumed that the configuration is specified at times t_1 and t_2 . For the cantilever beam, this means that

$$\delta u(x, t_1) = \delta u(x, t_2) = \delta v(x, t_1) = \delta v(x, t_2) = 0 \quad (\text{B-11})$$

which takes care of the first four conditions above.

Since δu , δv are not arbitrarily equal to zero, it follows from eq. (B-10v) and (B-10vi) that:

$$\rho \cdot u_{tt} - E \cdot A \cdot (u' + 0.5 \cdot v'^2)' = 0 \quad (\text{B-12})$$

and

$$\rho \cdot v_{tt} + E \cdot I \cdot v'''' - E \cdot A \cdot [(u' + 0.5 \cdot v'^2) \cdot v']' = 0 \quad (\text{B-13})$$

which are the equations of motion for free axial and transverse vibrations of the beam.

The other conditions are the natural and geometric boundary conditions for the cantilever beam.

At the fixed end ($x=0$), δu and δv are zero, therefore:

(from B-10xii and B-10x)

$$u(0,t) = 0 \quad (B-14)$$

and $v(0,t) = 0.$ (B-15)

At the free end ($x=L$), δu and δv are not necessarily zero, hence:

(from B-10xi and B-10ix)

$$E \cdot A \cdot (u' + 0.5 \cdot v'^2) = 0 \text{ or } u' + 0.5 \cdot v'^2 = 0 \text{ at } x=L \quad (B-16)$$

and

$$- E \cdot I \cdot v''' + E \cdot A \cdot (u' + 0.5 \cdot v'^2) \cdot v' = 0 \text{ at } x=L \quad (B-17a)$$

but substituting the result from equation B-16 into equation B-17a, we get:

$$v'''(L,t) = 0 \quad (B-17b)$$

Finally, at the fixed end the slope $\delta v'$ is zero and at the free end, the slope is not necessarily equal to zero. Therefore:

(from eq. B-10vii and B-10viii)

$$v'(0,t) = 0 \quad (B-18)$$

and

$$v''(L,t) = 0 \quad (B-19)$$

We may linearize the equations of motion by striking out the nonlinear term $0.5 \cdot v'^2$ to obtain a linear differential equation with linear boundary conditions for the axial displacement $u(x,t)$. This results in the following equations:

$$\rho \cdot u_{tt} - E \cdot A \cdot u'' = 0 \quad (B-12a)$$

and

$$\rho \cdot v_{tt} + E \cdot I \cdot v'''' - E \cdot A \cdot [u' \cdot v']' = 0 \quad (B-13a)$$

$$u(0,t) = 0 \quad (B-14)$$

$$v(0,t) = 0. \quad (B-15)$$

$$u'(L,t) = 0 \quad (B-16a)$$

$$v'''(L,t) = 0 \quad (B-17b)$$

$$v'(0,t) = 0 \quad (B-18)$$

$$v''(L,t) = 0 \quad (B-19)$$

Furthermore, the equation of motion for the transverse vibration may be linearized by striking out the nonlinear terms $u'' \cdot v'$ and $u' \cdot v''$ which results in the following form:

$$\rho \cdot v_{tt} + E \cdot I \cdot v'''' = 0 \quad (B-13b)$$

The equations of motion are now uncoupled and they can be solved separately.

Assuming harmonic motion for the free transverse vibration of the beam given by the equation:

$$v(x,t) = \phi(x) \cdot v(t), \quad (B-20)$$

where

$$v(t) = \cos(\omega t - \alpha)$$

and substituting this into eq. (B-13b), we get the eigenvalue equation

$$E \cdot I \cdot d^4 \phi / dx^4 - \rho \cdot \omega^2 \cdot \phi = 0 \quad (B-21)$$

Closed-form solutions are not available for this equation with variable coefficients.

For a uniform beam, eq. (B-21) can be written as:

$$d^4 \phi / dx^4 - \lambda^4 \cdot \phi = 0 \quad (B-22)$$

where

$$\lambda^4 = (\rho \cdot \omega^2) / (E \cdot I) \quad (B-23)$$

Let $\phi(x) = e^{nx}$ and substitute it into eq. (B-22) to get:

$$e^{nx} \cdot (n^4 - \lambda^4) = 0. \quad (B-24)$$

The values of n that satisfy the above equation are found to be:

$$n_1 = \lambda$$

$$n_2 = -\lambda$$

$$n_3 = \lambda i$$

$$\text{and } n_4 = -\lambda i$$

where i is the imaginary part symbol ($i = [-1]^{0.5}$).

The general solution of equation (B-22) may be written in several forms, such as:

$$\phi(x) = A_1 \cdot e^{\lambda x} + A_2 \cdot e^{-\lambda x} + A_3 \cdot e^{\lambda x i} + A_4 \cdot e^{-\lambda x i} \quad (\text{B-25a})$$

$$\text{or } \phi(x) = B_1 \cdot e^{\lambda x} + B_2 \cdot e^{-\lambda x} + B_3 \cdot \sin(\lambda x) + B_4 \cdot \cos(\lambda x) \quad (\text{B- 25b})$$

$$\text{or } \phi(x) = C_1 \cdot \sinh(\lambda x) + C_2 \cdot \cosh(\lambda x) + C_3 \cdot \sin(\lambda x) + C_4 \cdot \cos(\lambda x). \quad (\text{B-25c})$$

Equation (B-25c) will be used in this report. This equation represents a typical normal (or eigen-) function for transverse vibrations of a prismatic beam. The constants C_1 , C_2 , C_3 , and C_4 are determined from the boundary conditions at the ends of the beam which were derived above in equations (B-15, B-17b, B-18, B-19).

At the fixed end ($x=0$), the deflection and slope are equal to zero:

$$\phi(0) = 0 \quad (\text{B-26})$$

$$\text{and } d\phi(0)/dx = 0. \quad (\text{B-27})$$

At the free end ($x=L$), the bending moment and the shear force both vanish:

$$d^2 \phi(L)/dx^2 = 0 \quad (\text{B-28})$$

and

$$d^3 \phi(L)/dx^3 = 0. \quad (\text{B-29})$$

From eq. (B-25c), we have:

$$d\phi/dx = \lambda[C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) + C_3 \cos(\lambda x) - C_4 \sin(\lambda x)] \quad (B-30)$$

$$d^2\phi/dx^2 = \lambda^2[C_1 \sinh(\lambda x) + C_2 \cosh(\lambda x) - C_3 \sin(\lambda x) - C_4 \cos(\lambda x)] \quad (B-31)$$

$$d^3\phi/dx^3 = \lambda^3[C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) - C_3 \cos(\lambda x) + C_4 \sin(\lambda x)]. \quad (B-32)$$

Substituting equation B-25c and equations B-30, B-31, B-32 into the boundary condition equations B-26, B-27, B-28, B-29, we get the following system of equations:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \lambda & 0 & \lambda & 0 \\ \lambda^2 \sinh \lambda L & \lambda^2 \cosh \lambda L & -\lambda^2 \sin \lambda L & -\lambda^2 \cos \lambda L \\ \lambda^3 \cosh \lambda L & \lambda^3 \sinh \lambda L & -\lambda^3 \cos \lambda L & \lambda^3 \sin \lambda L \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (B-33)$$

This homogeneous system of equations will have nontrivial solutions if the determinant of the coefficient matrix is zero. Setting this determinant to zero yields the characteristic equation for the system:

$$\cos \lambda L * \cosh \lambda L + 1 = 0 \quad (B-34)$$

whose roots are the eigenvalues $\lambda_i L$ and i represents the i -th root. A numerical solution of equation (B-34) is required. The first four

consecutive roots of this equation are:

$$\begin{aligned}\lambda_1 L &= 1.8751 & \lambda_2 L &= 4.6941 \\ \lambda_3 L &= 7.8548 & \text{and } \lambda_4 L &= 10.99554.\end{aligned}\tag{B-35}$$

From equation (B-23),

$$\omega_i = [(\lambda_i L)^2 / L^2] * [E * I / \rho]^{0.5}\tag{B-36}$$

so

$$\omega_1 = (3.516 / L^2) * [E * I / \rho]^{0.5}\tag{B-37}$$

$$\omega_2 = (22.03 / L^2) * [E * I / \rho]^{0.5}\tag{B-38}$$

$$\omega_3 = (61.70 / L^2) * [E * I / \rho]^{0.5}\tag{B-39}$$

$$\omega_4 = (120.90 / L^2) * [E * I / \rho]^{0.5}\tag{B-40}$$

The normal function or eigenfunction for the cantilevered beam is found as follows:

$$\phi_i(x) = (\cosh \lambda_i x - \cos \lambda_i x) - \alpha_i (\sinh \lambda_i x - \sin \lambda_i x)\tag{B-41}$$

where

$$\alpha_i = (\cosh \lambda_i L + \cos \lambda_i L) / (\sinh \lambda_i L + \sin \lambda_i L).\tag{B-42}$$

Using the roots of the frequency equation, we get:

$$\begin{aligned}\lambda_1 L &= 1.8751 & \alpha_1 &= 0.7340955 \\ \lambda_2 L &= 4.6941 & \alpha_2 &= 1.0184673 \\ \lambda_3 L &= 7.8548 & \alpha_3 &= 0.9992244 \\ \lambda_4 L &= 10.99554 & \alpha_4 &= 1.0000336.\end{aligned}\tag{B-43}$$

The first four eigenfunctions may be written as:

$$\begin{aligned}\phi_1(x) &= 1.0 \cdot (\cosh 1.8751x/L - \cos 1.8751x/L) \\ &- 0.7340955 \cdot (\sinh 1.8751x/L - \sin 1.8751x/L),\end{aligned}\tag{B-44}$$

$$\begin{aligned}\phi_2(x) &= 1.0 \cdot (\cosh 4.6941x/L - \cos 4.6941x/L) \\ &- 1.0184673 \cdot (\sinh 4.6941x/L - \sin 4.6941x/L),\end{aligned}\tag{B-45}$$

$$\begin{aligned}\phi_3(x) &= 1.0 \cdot (\cosh 7.8548x/L - \cos 7.8548x/L) \\ &- 0.9992244 \cdot (\sinh 7.8548x/L - \sin 7.8548x/L),\end{aligned}\tag{B-46}$$

$$\begin{aligned}\phi_4(x) &= 1.0 \cdot (\cosh 10.99554x/L - \cos 10.99554x/L) \\ &- 1.0000336 \cdot (\sinh 10.99554x/L - \sin 10.99554x/L)\end{aligned}\tag{B-47}$$

The general solution of an N-term approximation for the transverse displacement $\phi(x)$ can be written as:

$$\phi(x) = \sum_{i=1}^N \phi_i(x)$$

The general solution of an N-term approximation of the transverse displacement $v(x,t)$ may be written as:

$$v(x,t) = \sum_{i=1}^N \phi_i(x) \cdot v_i(t)\tag{3.2-a}$$

Similarly, for free axial vibration of the cantilevered beam, we may assume harmonic motion given by:

$$u(x,t) = \psi(x) \cdot u(t) \quad (B-48)$$

where

$$u(t) = \cos(\omega \cdot t - \alpha).$$

Substituting this expression into equation (B-12a), we get the eigenvalue equation:

$$d[E \cdot A \cdot (d\psi/dx)]/dx + \rho \cdot \omega^2 \cdot \psi = 0 \quad (B-49)$$

For a uniform beam, the above equation can be written as:

$$d^2 \psi/dx^2 + [\rho \cdot \omega^2 / (E \cdot A)] \cdot \psi = 0 \quad (B-50)$$

or as

$$d^2 \psi/dx^2 + \lambda^2 \cdot \psi = 0 \quad (B-51)$$

where

$$\lambda^2 = \rho \cdot \omega^2 / (E \cdot A) \quad (B-52)$$

or

$$\omega = \lambda \cdot [E \cdot A / \rho]^{0.5} \quad (B-52a)$$

The general solution of equation (B-51) is

$$\psi(x) = A_1 \cdot \cos (\lambda \cdot x) + A_2 \cdot \sin (\lambda \cdot x) \quad (B-53)$$

The boundary conditions were derived in equations (B-14) and (B-16a) which can be written here as:

$$\psi(0) = 0 \quad (B-54)$$

and

$$d \psi(L)/dx = 0 \quad (B-55)$$

Differentiating eq. (B-53) with respect to x , we get:

$$d \psi/dx = -A_1 \lambda \sin (\lambda x) + A_2 \lambda \cos (\lambda x) \quad (B-56)$$

Substituting the boundary conditions into equations (B-53) and (B-56), we get:

$$A_1 = 0 \quad (B-57)$$

$$A_2 \lambda \cos (\lambda L) = 0 \quad (B-58)$$

To get a nontrivial solution, we must select a value of λ such that

$$\cos (\lambda L) = 0. \quad (B-59)$$

This is the characteristic equation for the free axial vibration of the cantilevered beam.

The roots of the characteristic equation are:

$$\lambda * L = \pi/2, 3\pi/2, \dots, [i - 0.5] * \pi, \dots \quad (B-60)$$

$$\text{or } (\lambda * L)_i = \lambda_i * L = \{ [i - 0.5] * \pi \} \quad \text{for } i = 1, 2, 3, \dots, N. \quad (B-60a)$$

$$\text{or } \lambda_i = \{ [i - 0.5] * \pi \} / L \quad \text{for } i = 1, 2, 3, \dots, N \quad (B-60b)$$

From equations (B-52a) and (B-60b), we write the natural frequencies as:

$$\omega_i = \lambda_i * [E * A / \rho]^{0.5} = \{ ([i - 0.5] * \pi) / L \} * [E * A / \rho]^{0.5} \quad (B-61)$$

or

$$\omega_i = \{ ([i - 0.5] * \pi) \} * [E * A / \rho * L^2]^{0.5} \quad (B-61a)$$

The corresponding mode shapes are obtained by combining equations (B-53) and (B-57) to get:

$$\psi_i(x) = A_2 * \sin(\lambda_i * x) = C * \sin [[i - .5] * \pi * (x/L)] \quad (B-62)$$

where A_2 is an arbitrary scaling factor.

The first four eigenfunctions for the free axial vibration are ($A_2 = 1$):

$$\psi_1(x) = 1.0 * \sin [.5 * \pi * (x/L)], \quad (B-63)$$

$$\psi_2(x) = 1.0 * \sin [1.5 * \pi * (x/L)], \quad (B-64)$$

$$\psi_3(x) = 1.0 \cdot \sin [2.5 \cdot \pi \cdot (x/L)], \quad (\text{B-65})$$

$$\psi_4(x) = 1.0 \cdot \sin [3.5 \cdot \pi \cdot (x/L)]. \quad (\text{B-66})$$

The general solution of an N-term approximation for the axial displacement $\psi(x)$ can be written as:

$$\psi(x) = \sum_{i=1}^N \psi_i(x)$$

The general solution of an N-term approximation of the axial displacement $u(x,t)$ may be written as:

$$u(x,t) = \sum_{i=1}^N \psi_i(x) \cdot u_i(t) \quad (3.2-a)$$

APPENDIX C

TABLE OF PHYSICAL CONSTANTS USED IN THE ANALYSIS:

(A) Stainless Steel Beam

$$E = 29 \cdot 10^6 \text{ psi}$$

$$\gamma_v = .283 \text{ lbf/in}^3$$

$$\rho_v = \gamma_v/g \text{ where } g = 386.4 \text{ in/s}^2 \text{ (acceleration of gravity)}$$

$$\rho_v = 0.7324 \cdot 10^{-3} \text{ lbf.s}^2/\text{in}^2$$

$$b = 1.0 \text{ in}$$

$$h = 0.001 \text{ in}$$

$$L = 0.250 \text{ in}$$

$$A = 0.001 \text{ in}^2$$

$$\rho = .7324 \cdot 10^{-6} \text{ lbf.s}^2/\text{in}^2$$

$$\rho = \text{mass per unit length} = \rho_v \cdot A$$

$$I = 0.8333 \cdot 10^{-10} \text{ in}^4$$

$$E \cdot I = 24.1657 \cdot 10^{-4} \text{ lbf.in}^2$$

$$E \cdot I/\rho = 3299.5 \text{ in}^4/\text{s}^2$$

$$E \cdot A = 29 \cdot 10^3 \text{ lbf}$$

$$E \cdot A/\rho = 3.959 \cdot 10^{10} \text{ in}^2/\text{s}^2$$

For longitudinal vibrations:

$$\lambda_1 \cdot L = \pi/2 = 1.5707963$$

$$\omega_1 = 1.24955 \cdot 10^6 \text{ rad/s}$$

$$\omega_2 = 3.74864 \cdot 10^6 \text{ rad/s}$$

(B) Polyurethane

$$E = 1000 \text{ psi}$$

$$\gamma_v = 0.04 \text{ lbf/in}^3$$

$$\rho_v = 0.10352 \cdot 10^{-3} \text{ lbf.s}^2/\text{in}^2$$

$$b = 1.0 \text{ in}$$

$$h = 0.050 \text{ in}$$

$$L = 0.250 \text{ in}$$

$$A = 0.050 \text{ in}^2$$

$$\rho = 0.5176 \cdot 10^{-5} \text{ lbf.s}^2/\text{in}^2$$

$$I = 104160 \cdot 10^{-10} \text{ in}^4$$

$$E \cdot I = 104.16 \cdot 10^{-4} \text{ lbf.in}^2$$

$$E \cdot I/\rho = 2012.4 \text{ in}^4/\text{s}^2$$

$$E \cdot A = 50 \text{ lbf}$$

$$E \cdot A/\rho = 9.66 \cdot 10^6 \text{ in}^2/\text{s}^2$$

$$\lambda_2 \cdot L = 3\pi/2 = 4.712389$$

$$\omega_1 = 19.518 \cdot 10^3 \text{ rad/s}$$

$$\omega_2 = 58.555 \cdot 10^3 \text{ rad/s}$$

For transverse vibrations:

$$\lambda_1 \cdot L = 1.8751$$

$$\omega_1 = 3229.75 \text{ rad/s}$$

$$\omega_2 = 20240.67 \text{ rad/s}$$

$$\lambda_2 \cdot L = 4.6941$$

$$\omega_1 = 2522.31 \text{ rad/s}$$

$$\omega_2 = 15807.205 \text{ rad/s}$$

$$N = .1 \text{ lb}$$

$$\mu_{\text{friction}} = 0.3 \text{ minimum}$$

$$0.4 \text{ maximum}$$

$$\theta = 30^\circ, 45^\circ, 60^\circ$$

$$v_w = 15.0 \text{ in/s}$$

$$\text{Roller Radius} = 1 \text{ in}$$

$$\text{Pertubation: } 1/\text{rev of roller}$$

$$\omega = 15 \text{ rad/s}$$

$$\omega \ll \omega_1$$

$$\beta = \gamma = 0^\circ$$

$$\mu = \mu_0 + \mu_1 \cdot \cos(\omega \cdot t) \text{ with } \mu_1 \ll \mu_0$$

$$N = .1 \text{ lb}$$

$$\mu_{\text{friction}} = 0.3 \text{ minimum}$$

$$1.5 \text{ maximum}$$

$$\theta = 30^\circ, 45^\circ, 60^\circ$$

$$v_w = 15.0 \text{ in/s}$$

$$\text{Roller Radius} = 1 \text{ in}$$

$$\text{Pertubation: } 1/\text{rev of roller}$$

$$\omega = 15 \text{ rad/s}$$

$$\omega \ll \omega_1$$

$$\beta = \gamma = 0^\circ$$

A FORTRAN computer program was written to calculate the coefficients a_1 and b_1 in Mathieu's equation based upon variations in the above parameters and with $\mu_1 = 0.01 \cdot \mu_0$.

REFERENCES

- [1] Guran, A., and Ossia, K., "On the Stability of a Flexible Missile under an End Thrust", Proceedings of the 7th Int. Conf. on Mathematical and Computer Modelling, Vol. 14, pp. 965-968, 1990, Great Britain.
- [2] Modi, C. O., "Nonlinear Dynamics of a Magnetoelastic System", M. Sc. Thesis, Rochester Institute of Technology, 1991.
- [3] Guran, A., and Rimrott, F. P. J. (1989a). "On the dynamic stability of an elastic rod under a slave tip loading", Proc. Twelvth Biennial Conf. on Mechanical Vibrations, ASME, 4, 225-228.
- [4] Mettler, E., "Dynamic Buckling", Handbook of Engineering Mechanics by W. Flugge, McGraw-Hill Book Co., 1962.
- [5] Timoshenko, S., and Gere, J., "Theory of Elastic Stability", McGraw-Hill Book Co., Inc., 1961.
- [6] Bolotin, V. V., "Dynamic Stability of Elastic Systems", Holden-Day Series in Mathematical Physics, Holden-Day, Inc., 1964.
- [7] Levinson, M., (1966) "Application of the Galerkin and Ritz Methods to Nonconservative Problems of Elastic Stability". ZAMP, 17, 431-442.
- [8] Leipholz, Horst, "Stability Theory, An Introduction to the Stability of Dynamic Systems and Rigid Bodies", John Wiley & Sons, 1987.
- [9] Nayfeh, Ali H., and Mook, Dean T., "Nonlinear Oscillations", John Wiley & Sons, 1979.
- [10] Cunningham, Walter J., "Introduction to Nonlinear Analysis", McGraw-Hill, 1958.
- [11] McLachlan, N. W., "Theory and Application of Mathieu Functions", Dover Publications, 1964.
- [12] Leccese, Andrew, "Stability of Time-Varying Linear Systems", M. Sc. Thesis, Rochester Institute of Technology, 1994.

- [13] Craig, R., Jr, "Structural Dynamics - An Introduction to Computer Methods", John Wiley and Sons, Inc., 1981.
- [14] Stoker, J. J., "Nonlinear Vibrations in Mechanical and Electrical Systems", Interscience Publishers, Inc., 1950.
- [15] Thomson, W. T., "Theory of Vibration with Applications", Prentice-Hall, Inc., 1981.
- [16] Chen, W. F., and Lui, E. M., "Structural Stability - Theory and Implementation", Chapter 6 - Energy and Energy Methods, Elsevier Science Publishing Co., Inc., 1987.
- [17] Libove, C., "Elastic Stability", Handbook of Engineering Mechanics by W. Flugge, McGraw-Hill Book Co., 1962.
- [18] Kuo, Benjamin C., "Automatic Control Systems", Third Edition, Prentice-Hall, Inc., 1975.
- [19] Reddy, J. N., "An Introduction to the Finite Element Method", McGraw-Hill Publishing Co., 1984.
- [20] Shames, I., and Dym, C., "Energy and Finite Element Methods in Structural Mechanics", Hemisphere Publishing Corp., 1985.
- [21] Char, B. W. et al (1989). First Leaves A Tutorial Introduction to Maple. Third Edition. Watcom Publications, Waterloo, Ontario, Canada.
- [22] Timoshenko, S., Young, D.H., and Weaver, W., "Vibration Problems in Engineering", John Wiley and Sons, Inc., 1990.
- [23] Spiegel, M. R., "Theoretical Mechanics with an Introduction to Lagrange's Equations and Hamiltonian Theory", Schaum's Outline series, McGraw-Hill Book Co., 1967.
- [24] The Student Edition of MATLAB for Macintosh Computers, The Math Works Inc., Prentice Hall, 1992.

- [25] “Lateral Vibrations of Beams: Classical Theory”, Handbook of Engineering Mechanics by W. Flugge, Section 61.4, McGraw-Hill Book Co., 1962.**
- [26] Seto, W. W., “Mechanical Vibrations”, Schaum’s Outline Series in Engineering, McGraw-Hill Book Co., 1964.**
- [27] Langhaar, H. L., “Energy Methods in Applied Mechanics”, John Wiley and Sons, Inc., 1962.**
- [28] Mantrala, Anand, “Transition Through Resonance in Linear and Nonlinear Systems”, M. Sc. Thesis, Rochester Institute of Technology, 1991.**
- [29] Advanced Continuous Simulation Language, Reference Manual, Edition 10.0, Mitchell & Gauthier Associates, 1991**

MAPLE COMPUTER PROGRAM USING GALERKIN'S METHOD

1 - Program to Input Eigenfunctions: (First two only)

```
# Francisco Ziegelmueller's Maple file to input eigenfunctions
# for the free vibration of a cantilever beam
# File name is y12.
# Eigenfunctions are to be inputted into GALERK2. file
#
# x is the coordinate of a point along the neutral axis of the beam
# l is the free extension of the beam
#
# Defining angles:
z1:= 1.8751*x/l;
z2:= 4.6941*x/l;
# Define the first two eigenfunctions:
y1:= 1.0000*( cosh(z1) - cos(z1) ) - 0.7340955*( sinh(z1) - sin(z1) );
y2:= 1.0000*( cosh(z2) - cos(z2) ) - 1.0184673*( sinh(z2) - sin(z2) );
#
# end of file y12.
```

2 - Maple file to compute GALERKIN's integral equations:

Program will need to have as input the following data:

n the number of eigenfunctions

y_i the eigenfunctions from file y12.

This is the program:

```
#
```

```

#   Francisco Ziegelmuller's Maple file named GALERK2.
#   File will compute Galerkin's integral equations using
#   the eigenfunctions for the free vibration of a cantilever
#   beam
#   The integrals will be stored as entries of a coefficient matrix
#   B of n rows and n columns
#   Each entry is a time differential equation for v(t) which will be
#   analyzed for the dynamic stability of the forced vibrations of the
#   beam.
#   Galerkin's method
#
#   Create a matrix B involving linear algebra operations
with(linalg):
b:=array(1..n,1..n);
#
#   define the functional for the problem :
#
func:= proc(y,y)
    int(( em*im*vtt*diff(y,x,x,x,x) + fx*v*diff(y,x,x) + RO*v*y)*y,
    x=0..l);
end;
#
#   calculate the entries of matrix B
for i from 1 to n do
    for j from 1 to n do
        b[i,j]:=expand( evalc (evalf (func (y.j, y.i)))));
    od;

```

```

od;
#
#   write out the determinant of matrix B :
#
detb:= expand(evalc(det(b))):
lprint(  ` Characteristic Equation `);
detb = 0;
#
#   Save the results into file named estavel.dat
#
save(b,detb,`estavel.dat`);
#   End of File

```


MATLAB PROGRAMS

The following MATLAB programs were provided by Dr. Joseph Torok to solve Mathieu's equation and determine the stability of the solutions:

MATLAB program POINTS.N

```
%  
%  
%          POINTS.M  
%  
%      First define      amax - the lower horizontal limit in parameter space  
%                        amin - the upper horizontal limit in the parameter space  
%                        bmin - the lower vertical limit in the parameter space  
%                        bmax - the upper vertical limit in the parameter space  
%  
%  
%          N - the GRID SIZE, i.e. number of points in each  
%              direction of the a-b plane.  
%  
delta_a= (amax-amin)/(N-1)  
delta_b= (bmax-bmin)/(N-1)  
%  
hor = (amin:delta_a:amax)  
%  
ver = (bmin:delta_b:bmax)  
%  
clear values  
clear S  
%
```

MATLAB program HILL.M

```
%  
%  
%  
%  
% This program defines Hill's equation  $y'' + (a + b \cdot q(t)) \cdot y = 0$   
% to be used in ode45 . The parameters are able to be  
% inputted (artificially) by setting them equal to  
%  $x(3) = a$  and  $x(4) = b$ , as "initial conditions".  
%  
function xdot = vce(T,x)  
% parametric oscillator  
xdot(1) = x(2);  
xdot(2) = -x(3)*x(1) - x(4)*cos(T)*x(1);  
xdot(3) = 0.0;  
xdot(4) = 0.0;  
end;
```

MATLAB program RUN.M

```
%  
% This program RUN.M calls the ODE-solver ode 45  
%  
%  
ti=0.0;  
tf=2*pi;  
%  
[t,x] = ode45('hill',ti,tf,xi);  
%
```

```
%
%
%
%   Note that it calls the defining function program HILL.M
%
%
```

MATLAB program RUNPTS.M

```
%
%
%   RUNPTS.M
%
%   This program actually determines the stability value
%
%    $S = \phi_1(2\pi) + \phi_2'(2\pi)$ 
%
%   It calls the program RUN.M, which solves for the fundamental
%   solutions, at each of the desired parameter values.
%
%   f1 and f2 represent the fundamental solutions.
%
values=zeros(N,N);
%
for i = 1:N
    for j = 1:N
        xi=[1,0,hor(i),ver(j)]';
        run
        f1=x(length(t),1);
        xi=[0,1,hor(i),ver(j)]';
```

```

run

f2=x(length(t),2);

values(i,j)=f1+f2;

end

end

%

% Rotate matrix to align horiz and vert
%

S=rot90(values);

%

% Once the program is run, the 'surface' S(a,b) can be
% plotted.
%
```

MATLAB program RUNPLT.M

```

%

% Program RUNPLT.M plots the stability surface
%

mesh(hor,ver,(values))

%

grid

title('Stability Surface for Mathieu Equation')

xlabel('a1 values')

ylabel('b1 values')

zlabel('S values')

%
```

MATLAB program CON.M

```
%  
%      CON.M  
%  
contour(hor,ver,values',[-2 2])  
grid  
xlabel('a1 values')  
ylabel('b1 values')  
title('Contour Plot for S=-2 or S=2')  
%
```

```

C PROGRAM TO CALCULATE THE COEFFICIENTS AT AND BT IN
C MATHIEU'S DIFFERENTIAL EQUATION FOR THE DYNAMIC
C STABILITY OF A CANTILEVER BEAM UNDER A PERIODICALLY
C VARYING LOAD
C MATHIEU'S EQ:  $V''(TAL) + (AT + BT \cdot \cos(TAL)) \cdot V(TAL) = 0$ 
C
C INPUT DATA:
C E..... BEAM ELASTIC MODULUS (PSI)
C XB..... BEAM WIDTH (IN)
C XL..... BEAM FREE LENGTH OR EXTENSION (IN)
C TH..... BEAM THICKNESS (IN)
C AREA..... BEAM CROSS-SECTIONAL AREA (IN^2)
C GAMMA..... BEAM MATERIAL VOLUMETRIC WEIGHT (LBF/IN^3)*
C NOTE: IN SEVERAL TEXTBOOKS, THE DENSITY IS CALLED UNIT WEIGHT
C AND LISTED AS LB/IN^3. IN USCS, THIS UNIT WEIGHT IS REALLY THE
C VOLUMETRIC MASS DENSITY IN LBM/IN^3. THIS LATTER VALUE IS WHAT
C WILL BE USED HERE FOR COMPUTATIONS.
C ROV..... VOLUMETRIC DENSITY OF MATERIAL (LBM/IN^3)
C RO..... BEAM MASS/UNIT FREE LENGTH:  $RO=ROV \cdot AREA$ 
C XI..... AREA MOMENT OF INERTIA
C THETA..... ANGLE BETWEEN BEAM AND ROLLER AT CONTACT
C EPI.....  $90 - THETA$ , EPIR IS EPI IN RADIANS
C XN..... NORMAL LOAD IN LBF
C COF0..... AVERAGE COEFFICIENT OF FRICTION
C COF1..... AMPLITUDE OF VARIATION OF COEF.OF FRICTION
C FX0..... COMPONENT OF FX RELATED TO COF0
C FX1..... COMPONENT OF FX RELATED TO COF1
C AT..... COEFFICIENT OF V IN MATHIEU'S EQUATION WHICH
C IS A CONSTANT (NOT DEPENDENT ON TAL),  $AT=A0+A1$ 
C A0..... A COMPONENT OF AT WHICH RELATES TO BEAM STIFFNESS
C A1..... COMPONENT OF AT WHICH RELATES TO FX0
C AW..... COEFFICIENT OF V IN MATHIEU'S EQUATION WHICH IS
C A CONSTANT ( =  $AT/W^2$ )
C BT..... A COMPONENT OF V WHICH VARIES PERIODICALLY AND
C WHICH RELATES TO FX1 TERM
C BW..... COEFFICIENT OF V IN MATHIEU'S EQUATION WHICH VARIES
C PERIODICALLY WITH TIME ( =  $BT/W^2$ )
C W..... FREQUENCY OF VARIATION IN FX1 (RAD/SEC)
C
C IMPLICIT REAL*8 (A-H, O-Z)
C OPEN (1,FILE="MAT")
C PI=3.141592654
C
C INPUT DATA FOR PROBLEM
C
C E=1000.0
C ROV=0.04
C W=15.0
C
C PERFORM AN ITERATION ON XL: FREE LENGTH
C
C DO 40 J=3,4
C XL=J*0.100 + 0.250
C XB=1.0
C TH=0.050
C AREA=XB*TH
C RO=ROV*AREA
C XI = XB*(TH**3)/12.0
C XN=0.10
C DO 40 I=2,4
C THETA=15.0*I
C EPI=90-THETA
C EPIR=EPI*PI/180.0

```

```

C
C      PERFORM AN ITERATION ON THE COFO
C
      DO 40 JJ=4,5
      COF0=0.3*JJ
      COF1=0.03
      FX0=XN*(COS(EPIR) + COF0*SIN(EPIR))
      FX1=COF1*XN*SIN(EPIR)
C
C      DO 40 I=1,2
C      IF (I .EQ. 1) THEN
      A0=12.35962*E*XI/(RO*(XL**4))
      A1=0.871558*FX0/(RO*(XL**2))
      AT=A0+A1
      BT=0.871558*FX1/(RO*(XL**2))
C
C      ELSE
C      A0=485.5251335*E*XI/(RO*(XL**4))
C      A1=-13.29431469*FX0/(RO*(XL**2))
C      AT=A0+A1
C      BT=-13.29431469*FX1/(RO*(XL**2))
C      ENDIF
      AW=AT/W**2
      BW=BT/W**2
C
C      PRINT OUT INPUT DATA
C
      WRITE(1,*)
      WRITE(1,100)
      WRITE(1,110)
      WRITE(1,120) E,ROV,XL,XB,TH
      WRITE(1,130)
      WRITE(1,120) AREA,RO,XI,XN,THETA
      WRITE(1,131)
      WRITE(1,120) EPI,COF0,COF1,FX0,FX1
      WRITE(1,132)
      WRITE(1,125) A0,A1,AT,AW
      WRITE(1,133)
40    WRITE(1,135) BT, BW
C
C      FORMAT STATEMENTS
C
100  FORMAT(10X,"CALCULATION OF PARAMETERS FOR MATHIEU'S EQUATION",/)
110  FORMAT(1X,"ELAST. MODULUS",4X,"DENSITY",8X,"FREE L",8X,"WIDTH"
1,9X,"THICK")
120  FORMAT(2X,5(E12.5,2X),/)
125  FORMAT(2X,4(E12.5,2X),/)
130  FORMAT(5X,"AREA",12X,"M/L",6X,"MOM. INERTIA",3X,"NORMAL LOAD",4X,
1"THETA(°)")
131  FORMAT(5X,"ETA(°)",9X,"COF0",11X,"COF1",10X,"FX0",11X,"FX1")
132  FORMAT(6X,"A0",14X,"A1",11X,"AT",12X,"A/W")
133  FORMAT(35X,"BT",12X,"B/W")
135  FORMAT(30X,2(E12.5,2X),/)
      CLOSE(1)
      STOP
      END

```